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Connected Certified Domination In The Middle Graph Of Certain Graphs

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Abstract

A dominating set S of a graph $G = (V, E)$ is called a certified dominating set of G. If every vertices in S has either zero or at least two neighbours in $V(G) - S$. A certified dominating set S of G is said to be connected certified dominating set if the subgraph induced by S is connected. The minimum cardinality taken over all the connected certified dominating set is called the connected certified domination number of G and is denoted by $\gamma_{cer}^{c}(G)$. in this paper, we investigate the connected certified domination number of middle graphs of certain graphs.

Keywords: Dominating set, certified dominating set, certified domination number, connected certified domination, Middle graphs.
AMS: 05C69

1. Introduction

Let $G = (V, E)$ be a finite, undirected graph without loops and multiple edges. The graph G has $n = |V|$ vertices and $m = |E|$ edges. A path P_n is a graph whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that the edges are $\{v_i v_{i+1}\}$, where $i = 1,2, \ldots, n-1$. A cycle is a path from a vertex back to itself.(So the first and last vertices are not distinct). A complete graph K_n is a graph in which any two distinct vertices are adjacent. A complete bipartite graph, denoted by $K_{m,n}$ is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y. A star is a complete bipartite graph $K_{1,n}$. The join $G + H$ of graphs G and H is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv; u \in V(G)\ and\ v \in V(H)\}$. The fan graph of order n is defined as $K_1 + P_n$ and is denoted by F_n or $F_{1,n}$. The wheel graph of order $n \geq 3$ is defined as $K_1 + C_n$ and is denoted by W_n or $W_{1,n}$. A vertex of degree 1 is called end vertex. A vertex which is adjacent to an end vertex is called support vertex. For more details we refer [1].
Domination in graphs is one of the interesting areas in graph theory which has wide applications in Engineering and Science. There are more than 300 domination parameters available in the literature. Around 1960 Berge and ore started the mathematical exploration of domination theory in graphs. There is a plethora of material on domination theory; we recommend readers outstanding books [2,3] on domination-related parameters.
Suppose that we are given a group of X officials and a group of Y civilians. There $x \in X$ for each civil $y \in Y$ who can attend x, and every time any such y is attending x, there must be also another civil $z \in Y$ that observes y. That is z must act as a kind of witness, to sidestep any mismanagement from y. In the case of a certain social network, what is the minimum number of connected officials necessary to ensure such a service? This aforementioned issue motivates us to propose the concept of connected certified domination.
The theory of certified domination was introduced by Dettlaff, Lemanska, Topp, Ziemann and Zylnski [9] and further studded in [8]. It has many applications in real life situations. The concept of connected certified domination was introduced by A. Ilyass and V.S.Goswami[10]. This motivated we to study the connected certified number in central graphs of certain standard graphs such as complete, complete bipartite graph, path graph, cycle graph, wheel graph, fan graph and double star graph.
In [9], authors studied certified domination number in graphs which is defined as follows:

Definition 1.1
Let $G = (V, E)$ be any graph of order n. A subset $S \subseteq V(G)$ is said to be a certified dominating set if G if S is a dominating set of G and every vertex in S has either zero or at least two neighbours in $V - S$. The certified domination number denoted by $\gamma_{cer}(G)$ is the minimum cardinality of certified dominating sets in G.

Definition 1.2.
Let $G=(V,E)$ be any connected graph of order n. A certified dominating set $S \subseteq V(G)$ is called a connected certified dominating set of G if its induced subgraph $G[S]$ is connected. The connected certified domination number is the minimum cardinality of a connected dominating set of G and we denoted it by $\gamma_{cer}^{c}(G)$.

2. Preliminaries

Theorem 2.1 [9] For any graph G of order $n \geq 2$, every certified dominating set of G contains its support vertices.
Theorem 2.2 [9] For any graph G of order n, $1 \leq \gamma_{cer}(G) \leq n$.
Observation 2.3 [10]
1) f $K_{m,n}$ be a complete bipartite graph, then $\gamma_{cer}^{c}(K_{m,n}) = 2$ for $3 \leq m \leq n$.
2) f $K_{1,n-1}$ be a star graph, then $\gamma_{cer}^{c}(K_{1,n}) = 1$ for $n \geq 2$.
3) f W_n be a wheel graph, then $\gamma_{cer}^{c}(W_n) = 1$.
4) f $S_{1,n,n}$ be a double star graph, then $\gamma_{cer}^{c}(S_{1,n,n}) = 2$, where $n \geq 2$.

Observation 2.4 [10]
1) f K_n s a complete graph, then $\gamma_{cer}^{c}(K_n) = 1$ for $n \geq 3$.
2) f P_n s a path graph, then $\gamma_{cer}^{c}(P_n) = n$ for $n \geq 4$.
3) f C_n s a cycle graph, then $\gamma_{cer}^{c}(C_n) = n$ for $n \geq 4$.
4) f F_n s a fan graph, then $\gamma_{cer}^{c}(F_n) = 1$ for $n \geq 3$.

Observation 2.5 [10]
For any connected graph G, $\gamma_{cer}(G) \leq \gamma_{cer}^{c}(G)$.

3. Meddle graphs

Danuta Michalak in [6] given an operation on a graph called as middle graphs and is given as below:
Definition 3.1 [6]
The middle graph of a graph $G=(V,E)$ is the graph $M(G)=(V \cup E, E_1)$, where $xy \in E_1$ if and only if either x and y are edges in G having a vertex in common or x is a vertex of G and y is an edge of G containing x.

Theorem 3.2
Let G be a connected graph of order $n \geq 2$. Then $1 \leq \gamma_{cer}^{c}(M(G)) \leq n + \binom{n}{2}$.
Proof.
Any single vertex of G need to be dominated all vertices of $M(G)$. So $\gamma_{cer}^{c}(M(G)) \geq 1$. Also the total number of vertices and edges of G can be dominate the vertices of $M(G)$ and is formed a connected certified dominating set of $M(G)$. Hence, $1 \leq \gamma_{cer}^{c}(M(G)) \leq n + \binom{n}{2}$.
Theorem 3.3
For any star graph $K_{1,n}$ of $n+1 \geq 2$, vertices $\gamma_{cer}^{c}\left(M(K_{1,n})\right) = n$.
Proof.
Let $x, x_1, x_2, \dots, x_n$ be the vertices of $K_{1,n}$ with central vertex x. Let $y_i = xx_i$, for $1 \leq i \leq n$. Then each $y_i (1 \leq i \leq n)$ is adjacent to both x and $x_i (1 \leq i \leq n)$. Also by the definition of $M(K_{1,n})$, each y_i is adjacent to y_j for $i \neq j$ and $1 \leq i,j \leq n$. Clearly $G[\{y_1, y_2, \dots, y_n\}]$ is a clique in $M(K_{1,n})$.

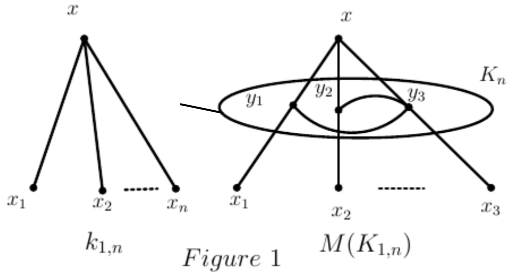
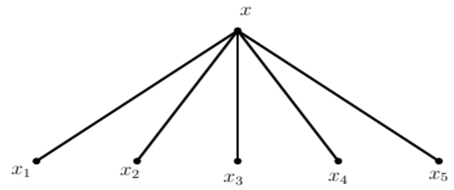


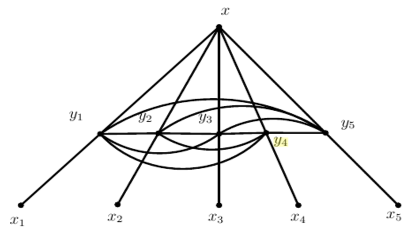
Figure 1

Here, $\left|V\left(M(K_{1,n})\right)\right| = 2n+1$. Since each $y_i(1 \le i \le n)$ is adjacent to the end vertices $x_i(1 \le i \le n)$, respectively in $M(K_{1,n})$ by theorem 2.1, $S = \{y_1, y_2, \ldots, y_n\}$ contained in every connected certified dominating set of $M(K_{1,n})$. Therefore, $\gamma_{cer}^{c}\left(M(K_{1,n})\right) \ge |S| = n$. Here, S dominates each $x_i(1 \le i \le n)$ and x. Also, every vertices in S has exactly two neighbours in $V\left(M(K_{1,n})\right) - S$. Therefore, that S itself a certified dominating set of G. Moreover, since $G[\{y_i\}]$ for all $i = 1,2,\ldots,n$ is complete, that $G[S]$ is connected. So S itself is a connected certified dominating set of G. Hence, $\gamma_{cer}^{c}\left(M(K_{1,n})\right) = n$.

Example 3.4.
For the graph $M(K_{1,5})$ in Figure 2, by Theorem 3.3, $S = \{y_1, y_2, y_3, y_4, y_5\}$ is a minimum connected certified dominating set of $M(K_{1,5})$ and hence, $\gamma_{cer}\left(M(K_{1,n})\right) = |S| = 5$.



$K_{1,5}$
Figure 2



$M(K_{1,5})$
Figure 3

Corollary 3.5
For the star graph $K_{1,n}, \gamma_{cer}^{c}\left(M(K_{1,n})\right) - \gamma_{cer}^{c}(K_{1,n}) = n-1$.
Proof.
The result follows from Theorem 3.3 and observation 2.3(2).
Theorem 3.6
For the cycle graph C_n of order $n \ge 3, \gamma_{cer}^{c}(M(C_n)) = n-1$.
Proof.
Let $x_1, x_2, \ldots, x_n$ be the vertices of C_n and $y_1, y_2, \ldots, y_n$ be the vertices added corresponding to the edges of $C_n$to form $M(C_n)$. Therefore, $|V(M(C_n)| = 2n$ and $\left|E\left(M(C_n)\right)\right| = 3n$.
In $M(C_n), y_1$ is adjacent to y_2, y_n, x_i and $x_2; y_i$ is adjacent to y_{i-1}, y_{i+1}, x_i and x_{i+1} for $2 \le i \le n-1$, y_n is adjacent to y_n and y_{n-1}, where $\deg(y_i) = 4$ and $\deg(x_i) = 2$ in $M(C_n)$. So $y_1, y_2, \ldots, y_n$ induces a cycle in $M(C_n)$ of length n. Thus every minimum connected certified dominating set must contains $y_1, y_2, \ldots, y_{n-1}$. So $\gamma_{cer}^{c}(M(C_n)) \ge n-1$.

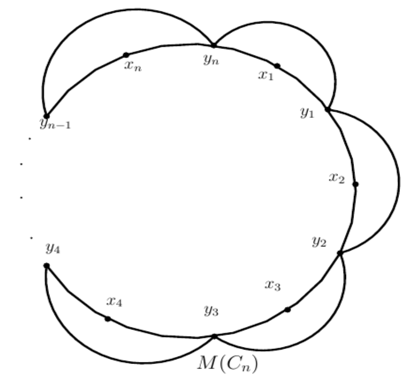


Figure 4

Now, starting from the vertex y_i for $1 \leq i \leq n, N[y_i] = \{x_i, x_{i+1}, y_i, y_{i-1}, y_{i+1}\}$ where suffices modulo n. Next we choose the vertex $y_{i+1}, N[y_{i+1}] = \{x_{i+1}, x_{i+2}, y_{i+1}, y_i, y_{i+1}\}$ proceeding like this, we get $S = \{y_1, y_2, \ldots, y_{n-1}\}$ is a connected certified dominating set of $M(C_n)$ and so $\gamma_{cer}^c(M(C_n)) \leq |S| = n - 1$. Hence, $\gamma_{cer}^c(M(C_n)) = n$.

Example 3.7.
Consider the graph $M(C_6)$ in Figure 5. By Theorem 3.6, $S = \{y_1, y_2, y_3, y_4, y_5\}$ is a minimum connected certified dominating set of $M(C_6)$ and hence, $\gamma_{cer}^c(M(C_6)) = 5$.

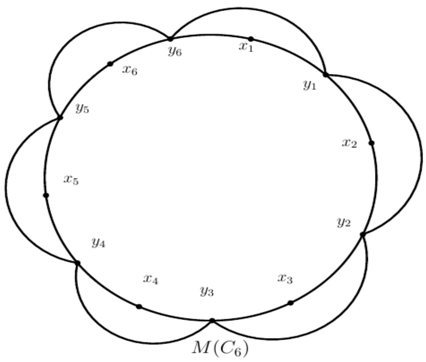
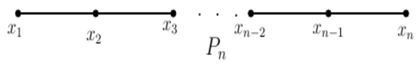


Figure 5

Theorem 3.8
For the path graph P_n of order $n \geq 2, \gamma_{cer}^c(M(P_n)) = n - 1$.
Proof.
Let $x_1, x_2, \ldots, x_n$ be the vertices of P_n and $y_1, y_2, \ldots, y_{n-1}$ be the vertices added in $M(P_n)$ corresponding to the edges of P_n. Therefore, clearly $|V(M(P_n))| = 2n - 1$ and $|E(M(P_n))| = 3n - 4$.



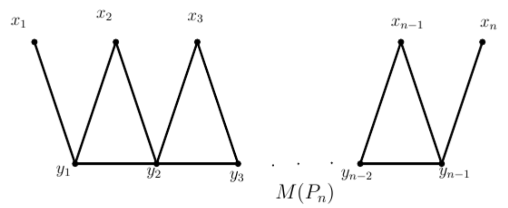


Figure 6

In $M(P_n), x_1$ is adjacent to y_1, x_n is adjacent to y_{n-1}, x_i is adjacent ot y_{i-1} and y_i for $2 \leq i \leq n-1$. Also, y_i is adjacent to x_1, x_2 and y_2, y_{n-1} is adjacent to x_{n-1}, x_n and y_{n-2}. Also, y_i is adjacent to y_{i-1}, y_{i+1} and x_{i+1} for $2 \leq i \leq n-2$.
Since x_1 and x_n are the pendent vertices adjacent to y_1 and y_{n-1}, respectively, by Theorem 2.1, every connected certified dominating set must contains y_1 and y_{n-1}. Since y_1 and y_{n-1} are connected by two distinct paths, we must need to select the shortest path that dominates all the vertices in $M(P_n)$. So every minimum connected certified dominating set of $M(P_n)$ must contains $y_1, y_2, \dots, y_{n-1}$. Therefore, $\gamma_{cer}^c(M(P_n)) \geq n-1$. Consider $S = \{y_1, y_2, \dots, y_{n-1}\}$. Clearly S dominates all the vertices in $M(P_n)$. Also, every vertices in S has exactly two neighbours in $V(M(P_n)) - S$. Therefore that S is a certified dominating set of $M(P_n)$. Furthermore $G[S]$ is a path in $M(P_n)$. So that S itself a connected certified dominating set of $M(P_n)$ and hence we conclude that $\gamma_{cer}^c(M(P_n)) = n-1$.

Example 3.9.

Consider the graph $M(P_7)$ in Figure 7, by Theorem 3.8, $S = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ is a minimum connected certified dominating set of $M(P_7)$ and hence, $\gamma_{cer}^c(M(P_7)) = 6$.

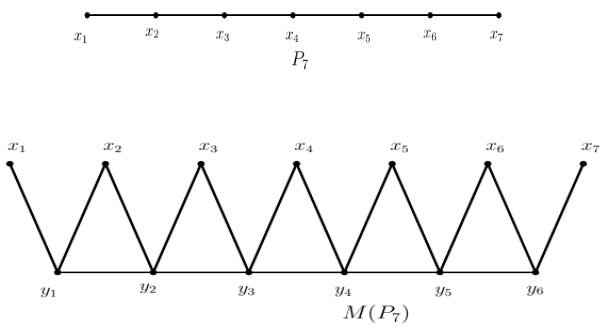


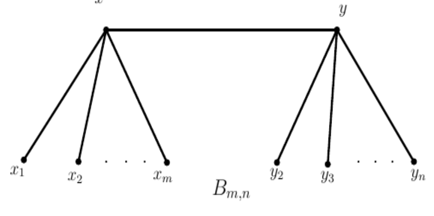
Figure 7

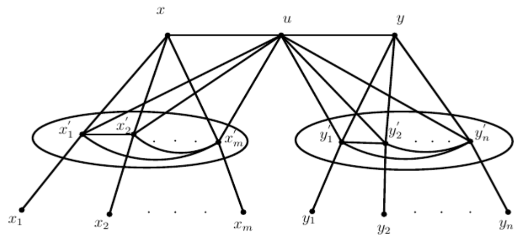
Theorem 3.10.

For the bistar graph $B_{m,n}, \gamma_{cer}^c(B_{m,n}) = m+n+1$.

Proof.

Consider the bistar graph $B_{m,n}$ with vertex set $V(B_{m,n}) = \{x, y, x_i, y_j\,; 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E(B_{m,n}) = \{xy, xx_i, yy_j; 1 \leq i \leq m, 1 \leq j \leq n\}$. Then by the definition of middle graph $x_1', x_2', \dots, x_m'$ and $y_1', y_2', \dots, y_n'$ be the vertices added corresponding to the edges $xx_1, xx_2, \dots, xx_m$ and $yy_1, yy_2, \dots, yy_n$, respectively. Also u be the new vertex corresponding to the edge xy in $M(B_{m,n})$. Clearly $|V(B_{m,n})| = 2m+2n+3$ and $E(B(m,n))| = 2 + \frac{m^2+n^2+5m+5n}{2}$.

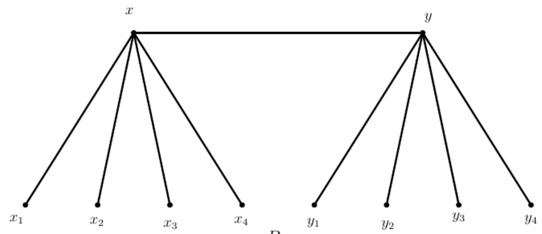




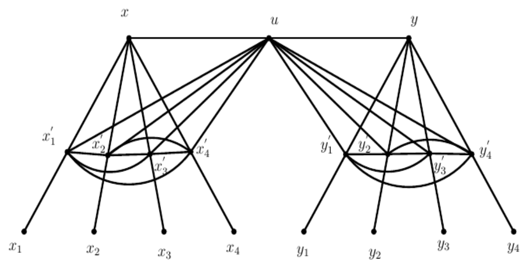
$M(B_{m,n})$
Figure 8

Here, each x_i' is adjacent to x, x_i and u for $1 \leq i \leq m$ and each $y_j'(1 \leq i \leq n)$. Clearly, the subgraph induced by $\{x_1', x_2', \dots, x_m'\}$ and $\{y_1', y_2', \dots, y_n'\}$ in $M(B_{m,n})$ are complete in $M(B_{m,n})$, respectively. Also, $S = \{x_1', x_2', \dots, x_m', y_1', y_2', \dots, y_n'\}$ be the set of support vertices in $M(B_{m,n})$. So by Theorem 2.1, every connected certified dominating set must contain S. Clearly, S dominates $V\left(M(B_{m,n})\right)$ and every vertices in S has exactly three neighbours in $V\left(M(B_{m,n})\right) - S$. Therefore that S itself for a certified dominating set of $M(B_{m,n})$. But the subgraph induced by S in $M(B_{m,n})$ is not connected. So that S is not a connected certified dominating set of $M(B_{m,n})$ and thus $\gamma_{cer}^c\left(M(B_{m,n})\right) \geq |S| + 1 = m + n + 1$. On the other hand, since u is adjacent to all the vertices in S, that $S \cup \{u\}$ form a connected certified dominating set of $M(B_{m,n})$ and hence $\gamma_{cer}^c\left(M(B_{m,n})\right) \leq |S \cup \{u\}| = m + n + 1$. Therefore, $\gamma_{cer}^c\left(M(B_{m,n})\right) = m + n + 1$.

Example 3.11.
For the graph $M(B_{4,4})$ in Figure 10, by Theorem 3.10, $S = \{u, x_1, x_2, x_3, x_7, y_1, y_2, y_3, y_4\}$ is a minimum connected certified dominating set of $M(B_{4,4})$ and hence, $\gamma_{cer}^c\left(M(B_{4,4})\right) = 9$.



$B_{4,4}$
Figure 9



$M(B_{4,4})$
Figure 10

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