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Some Generalized Results On Common Fixed Point In Vector Valued Rectangular Metric Spaces

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Abstract

The concept of rectangular metric space was initially introduced by Branciari [6] in 2000. This paper aims to demonstrate various common fixed point results for vector valued rectangular metric spaces. Our findings extend and certain established results in the scalar valued case. We also give some examples to illustrate our work.

Keywords: Weakly compatible, Archimedean, Riesz space, Common fixed point, Vector valuedrectangular metric spaces.

1. Introduction

A fixed point of a function is a point in the domain of the function that maps to itself under the function. In the past several years, fixed point theory has become widely recognized as a potent and essential tool in the exploration of nonlinear analysis. Stefa Banach [5] proved a fundamental result in the study of metric space called Banach fixed-point theorem in 1922, which state that "Every contraction in a complete metric space have a unique fixed point". Let S and T be two self maps on a metric space(Z,κ). If there exists a point $\varsigma \in Z$ such that $S\varsigma = T\varsigma = \varsigma$ then ς is common fixed point (CFP) of S and T. The investigation of CFPs for functions under specific circumstances has been a focal point of robust observation and several influential outcomes have been well-established by multiple authors. Jungck [10] presented a result on CFPs for commuting maps, offering a broader perspective that goes beyond the Banach fixed point theorem. Čuneyt Cevik and Ishak Altun [7] proposed a generic metric space called vector metric space, which is characterized by its Riesz space valued metric. Branciari [6] introduced the definition of rectangular metric space and derived various fixed-point results on this space. Using the concepts of vector metric space and rectangular metric space, we introduce vector valued rectangular metric space and prove CFP results for the same.

2. Preliminaries

For basic concepts and results regarding Riesz space, one may refer to C. D. Aliprantis and K. C. Border [2]. A partially ordered set is considered a lattice if for any two elements within the set, there exists both an infimum and a supremum. A Riesz space $\mathbb{V}$ is defined as a partially ordered vector space that also forms a lattice under its specified ordering. In Riesz space $\mathbb{V}$, consider $\ell \in \mathbb{V}$ and define the following:
$\ell^+ = \ell \vee 0, \ell^- = (-\ell) \vee 0$ and $|\ell| = \ell \vee (-\ell)$.
The notation $\vartheta_m \downarrow \vartheta$ signifies that $\{\vartheta_m\}$ is a decreasing sequence in Riesz space $\mathbb{V}$ and infimum of ϑ_m is ϑ. Also, $\mathbb{V}$ is an $\mathbb{V}$-Archimedean if $\frac{1}{m}\vartheta \downarrow 0$ for each $\vartheta \in \mathbb{V}_+ = \{\vartheta \in \mathbb{V} : 0 \leq \vartheta\}$.

Lemma 2.1. [3] Let $\mathbb{V}$ be a Riesz space and $\vartheta \leq a\vartheta$, $\forall\, \vartheta \in \mathbb{V}_+$ also $0 \leq a < 1$, then $\vartheta = 0$.
Definition 2.2. [3] Let Z be any non-empty set and $\mathbb{V}$ be a Riesz space. Then the mapping
$\kappa : Z \times Z \to \mathbb{V}$ is vector metric if it possesses the properties listed below:
(a) $\kappa(\vartheta,\omega) = 0 \Leftrightarrow \vartheta = \omega$
(b) $\kappa(\vartheta,\omega) = \kappa(\omega,\vartheta)$
(c) $\kappa(\vartheta,\omega) \leq \kappa(\vartheta,\eta) + \kappa(\eta,\omega),\ \forall\, \eta,\vartheta,\omega \in Z$
Then $(Z,\kappa,\mathbb{V})$ is called vector metric space(denoted as VMS).
Example 2.3. [3] Let $\mathbb{V} = R^2$ and a map $\kappa : R \times R \to \mathbb{V}$ defined as:

$$\kappa(\vartheta,\omega) = (o_1|\vartheta - \omega|, o_2|\vartheta - \omega|)$$

where $0 < o_1 + o_2$ and $0 \leq o_1, o_2$. Then the triplet $(R,\kappa,\mathbb{V})$ is VMS.
Definition 2.4. [3] A sequence $\{\vartheta_m\}$ in VMS $(Z,\kappa,\mathbb{V})$ is called vectorial convergent($\mathbb{V}$-convergent) to some $\vartheta \in \mathbb{V}$, if

$\exists\ r_m$ in $\mathbb{V}$ s.t. $r_m \downarrow 0$ and $\kappa(\vartheta_m, \vartheta) \leq r_m$ and denoted as $\vartheta_m \overset{\kappa,Q}{\rightarrow} \vartheta$. A sequence $\{\vartheta_m\}$ is $\mathbb{V}$-Cauchy, if $\exists\ r_m$ in $\mathbb{V}$ s.t. $r_m \downarrow 0$ and $\forall\ m, p$, we have $\kappa(\vartheta_m, \vartheta_{m+p}) \leq r_m$. If every $\mathbb{V}$-Cauchy sequence in Z is $\mathbb{V}$-convergent to a limit in Z, then VMS Z is said to be $\mathbb{V}$-complete.

Definition 2.5. [6] If Z is any non-empty set. The function $\kappa : Z \times Z \rightarrow R$ s.t. $\forall\ \vartheta, \omega \in Z$ and $\forall\ \eta, \varsigma \in Z$ each distinct from ς and η s.t. $\eta, \varsigma \notin \{\vartheta, \omega\}$ is called rectangular metric if it satisfies the following:

(a) $\kappa(\vartheta, \omega) = 0 \Leftrightarrow \vartheta = \omega$

(b) $\kappa(\vartheta, \omega) = \kappa(\omega, \vartheta)$

(c) $\kappa(\vartheta, \omega) \leq \kappa(\vartheta, \eta) + \kappa(\eta, \varsigma) + \kappa(\varsigma, \omega)$

Then (Z, κ) is termed a rectangular metric space (RMS).

Below we give an example of RMS which is not a metric space.

Example 2.6. [4] Let $Z = R$, $0 < r \in Z$ and define $\kappa : Z \times Z \rightarrow R$ s.t. for all $\vartheta, \omega \in Z$, $\kappa(\vartheta, \omega) = \kappa(\omega, \vartheta)$ and

$$\kappa(\vartheta, \omega) = \begin{cases} 3r & \text{if } (\vartheta, \omega) = (1,2) \text{ or } (2,1) \\ 0 & \text{if } \vartheta = \omega, \\ r & \text{if } (\vartheta, \omega) \neq (1,2) \text{ or } (2,1) \end{cases}$$

Since $3r = \kappa(1,2) > \kappa(1,3) + \kappa(3,2) = r + r$, hence space (Z, κ) is not metric space but RMS.

Definition 2.7. [13] Let Z be any non-empty set and $\mathbb{V}$ be a Riesz space. A vector valued rectangular metric is a mapping $\kappa : Z \times Z \rightarrow \mathbb{V}$ if $\forall\ \vartheta, \omega \in Z$ and $\forall\ \eta, \varsigma \in Z$ each distinct from ς and η s.t. $\eta, \varsigma \notin \{\vartheta, \omega\}$, it satisfies the following:

(a) $\kappa(\vartheta, \omega) = 0 \Leftrightarrow \vartheta = \omega$

(b) $\kappa(\vartheta, \omega) = \kappa(\omega, \vartheta)$

(c) $\kappa(\vartheta, \omega) \leq \kappa(\vartheta, \eta) + \kappa(\eta, \varsigma) + \kappa(\varsigma, \omega)$.

Then the triplet $(Z, \kappa, \mathbb{V})$ is vector valued rectangular metric space (denoted as VVRMS).

Next, we illustrate that a VVRMS need not be VMS by the following example.

Example 2.8. [13] Let $\mathbb{V} = R^2$, $Z = \{\vartheta : \vartheta$ is an integer and $0 \leq \vartheta \leq 3\}$ and $\forall\ \vartheta, \omega \in Z$ define mapping $\kappa : Z \times Z \rightarrow \mathbb{V}$ as $\kappa(\vartheta, \omega) = \kappa(\omega, \vartheta)$ and

$$\kappa(\vartheta, \omega) = \begin{cases} (0,0) & \text{if } \vartheta = \omega, \\ (3,3) & \text{if } (\vartheta, \omega) = (2,3) \text{ or } (3,2) \\ (1,1) & \text{if } (\vartheta, \omega) \neq (2,3) \text{ or } (3,2) \end{cases}$$

Since $\kappa(0,3) + \kappa(2,0) = (1,1) + (1,1) \prec \kappa(2,3) = (3,3)$, hence $(Z, \kappa, \mathbb{V})$ is not VMS but VVRMS.

Definition 2.9. [11] Let $S, T : Z \rightarrow Z$. If $\varsigma = S(\vartheta) = T(\vartheta)$ for any $\vartheta \in Z$, In this case ϑ is known as a coincidence point of S and T, and ς as a point of coincidence (PoC) for both S and T.

Definition 2.10. [11] The self-mappings $S, T : Z \rightarrow Z$ are known as weakly compatible (WC) if $S(\vartheta) = T(\vartheta)$ for any $\vartheta \in Z$, implies $S(T(\vartheta)) = T(S(\vartheta))$.

Proposition 2.11. [1] If two WC maps T and S defined on Z, possess a unique PoC denoted by $\varsigma = T\vartheta = S\vartheta$, then ϑ serves as the unique CFP for both T and S.

Proof. Given that $\varsigma = T\vartheta = S\vartheta$ and T and S are WC, then we have $T\varsigma = T(S\vartheta) = S(T\vartheta) = S\varsigma$, i.e., $T\varsigma = S\varsigma$ is a PoC of T and S. Although ς is the only PoC of T and S, so $\varsigma = T\varsigma = S\varsigma$. Hence ς is a CFP. To establish the uniqueness of CFP ς, let $\beta \in Z$ s.t. $\beta = T(\beta) = S(\beta)$. Then β is a PoC for both T and S. Since PoC is unique and therefore $\beta = \varsigma$. Thus ς is the unique CFP for both S and T.

3. Main Results

Inspired by the results of Kamra et al.[12], Rad et al. [16] and George et al.[8], we demonstrate certain CFP results in VVRMS.

Theorem 3.1 Let $(Z, \kappa, \mathbb{V})$ be a VVRMS with $\mathbb{V}$-Archimedean and the self mappings S and T on Z satisfies the following conditions:

(i) $\forall\ \varsigma, \vartheta \in Z, \kappa(T\varsigma, T\vartheta) \leq \gamma H(\varsigma, \vartheta)$

where $\gamma \in [0,1)$, $2 \leq a$ is a constant and

$H(\varsigma, \vartheta) \in \{\kappa(S\varsigma, S\vartheta), \kappa(S\varsigma, T\varsigma), \kappa(S\vartheta, T\vartheta), \frac{1}{a}[\kappa(S\varsigma, T\varsigma) + \kappa(S\vartheta, T\vartheta)]\}$

(ii) $T(Z) \subseteq S(Z)$

(iii) Subspace $T(Z)$ or $S(Z)$, is $\mathbb{V}$ -complete.

Then S and T possess a PoC which is unique in Z. If we assume WC of S and T then there exists a unique CFP of S and T.

Proof. Fix arbitrary $\varsigma_0 \in Z$. Define the sequence $\{\vartheta_m\}$ by $S\varsigma_{m+1} = T\varsigma_m = \vartheta_m$ where $m \geq 0$. Then

$$\kappa(\vartheta_m, \vartheta_{m+1}) = \kappa(T\varsigma_m, T\varsigma_{m+1}) \le \gamma H(\varsigma_m, \varsigma_{m+1}), \quad (1)$$

where

$$H(\varsigma_m, \varsigma_{m+1}) \in \{ \kappa(S\varsigma_m, S\varsigma_{m+1}), \kappa(S\varsigma_m, T\varsigma_m), \kappa(S\varsigma_{m+1}, T\varsigma_{m+1}), \tfrac{1}{a} [\kappa(S\varsigma_m, T\varsigma_m) + \kappa(S\varsigma_{m+1}, T\varsigma_{m+1}] \}$$

$$= \left\{\kappa(\vartheta_{m-1}, \vartheta_m), \kappa(\vartheta_{m-1}, \vartheta_m), \kappa(\vartheta_m, \vartheta_{m+1}), \tfrac{1}{a}[\kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\vartheta_m, \vartheta_{m+1})]\right\}$$

$$= \left\{\kappa(\vartheta_{m-1}, \vartheta_m), \kappa(\vartheta_m, \vartheta_{m+1}), \frac{1}{a}[\kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\vartheta_m, \vartheta_{m+1})]\right\}.$$

The possible three cases are:

(i) If $H(\varsigma_m, \varsigma_{m+1}) = \kappa(\vartheta_{m-1}, \vartheta_m)$, implies $\kappa(\vartheta_m, \vartheta_{m+1}) \le \gamma\kappa(\vartheta_{m-1}, \vartheta_m)$.

(ii) If $H(\varsigma_m, \varsigma_{m+1}) = \kappa(\vartheta_m, \vartheta_{m+1})$, we get

$$\kappa(\vartheta_m, \vartheta_{m+1}) \le \gamma\kappa(\vartheta_m, \vartheta_{m+1}).$$

Implies $\kappa(\vartheta_m, \vartheta_{m+1}) = 0$.

(iii) If $H(\varsigma_m, \varsigma_{m+1}) = \frac{1}{a}[\kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\vartheta_m, \vartheta_{m+1})]$. Then we have

$$\kappa(\vartheta_m, \vartheta_{m+1}) \le \frac{\gamma}{a}[\kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\vartheta_m, \vartheta_{m+1})]$$

$$\left(1 - \frac{\gamma}{a}\right) \kappa(\vartheta_m, \vartheta_{m+1}) \le \frac{\gamma}{a}\kappa(\vartheta_{m-1}, \vartheta_m)$$

$$\kappa(\vartheta_m, \vartheta_{m+1}) \le \left(\frac{\gamma}{a-\gamma}\right)\kappa(\vartheta_{m-1}, \vartheta_m).$$

Thus

$$\kappa(\vartheta_m, \vartheta_{m+1}) \le \lambda\, \kappa(\vartheta_{m-1}, \vartheta_m), \quad (2)$$

where $\lambda \in \left\{\gamma, \frac{\gamma}{a-\gamma}\right\} < 1$. Repeating the process of (2), we get

$$\kappa(\vartheta_m, \vartheta_{m+1}) \le \lambda^m\, \kappa(\vartheta_0, \vartheta_1). \quad (3)$$

Now

$$\kappa(\vartheta_m, \vartheta_{m+2}) = \kappa(T\varsigma_m, T\varsigma_{m+2}) \le \gamma H(\varsigma_m, \varsigma_{m+2}),$$

where

$$H(\varsigma_m, \varsigma_{m+2}) \in \{\kappa(S\varsigma_m, S\varsigma_{m+2}), \kappa(S\varsigma_m, T\varsigma_m), \kappa(S\varsigma_{m+2}, T\varsigma_{m+2}), \frac{1}{a}[\kappa(S\varsigma_m, T\varsigma_m) + \kappa(S\varsigma_{m+2}, T\varsigma_{m+2})]\}$$

$$= \{ \kappa(\vartheta_{m-1}, \vartheta_{m+1}), \kappa(\vartheta_{m-1}, \vartheta_m), \kappa(\vartheta_{m+1}, \vartheta_{m+2}), \frac{1}{a} [\kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\vartheta_{m+1}, \vartheta_{m+2})]\}.$$

The possible cases are:

(i) If $H(\varsigma_m, \varsigma_{m+2}) = \kappa(\vartheta_{m-1}, \vartheta_{m+1})$, then we have

$$\kappa(\vartheta_m, \vartheta_{m+2}) \le \gamma\, \kappa(\vartheta_{m-1}, \vartheta_{m+1})$$

$$\le \gamma[\kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\vartheta_m, \vartheta_{m+2}) + \kappa(\vartheta_{m+2}, \vartheta_{m+1})]$$

$$(1 - \gamma)\kappa(\vartheta_m, \vartheta_{m+2}) \le \gamma[\lambda^{m-1}\kappa(\vartheta_0, \vartheta_1) + \lambda^{m+1}\kappa(\vartheta_0, \vartheta_1)] \text{ (using (3))}$$

$$\kappa(\vartheta_m, \vartheta_{m+2}) \le \lambda^{m-1} \left[\frac{\gamma(1+\lambda^2)}{1-\gamma}\right]\kappa(\vartheta_0, \vartheta_1).$$

(ii) If $H(\varsigma_m, \varsigma_{m+2}) = \kappa(\vartheta_{m-1}, \vartheta_m)$, then we have

$$\kappa(\vartheta_m, \vartheta_{m+2}) \le \gamma\, \kappa(\vartheta_{m-1}, \vartheta_m)$$

$$\le \gamma\lambda^{m-1}\kappa(\vartheta_0, \vartheta_1) \text{ (using (3))}.$$

(iii) If If $H(\varsigma_m, \varsigma_{m+2}) = \kappa(\vartheta_{m+1}, \vartheta_{m+2})$, then we have

$$\kappa(\vartheta_m, \vartheta_{m+2}) \le \gamma\, \kappa(\vartheta_{m+1}, \vartheta_{m+2})$$

$$\le \gamma\lambda^{m+1}\kappa(\vartheta_0, \vartheta_1) \text{ (using (3))}.$$

(iv)) If If $H(\varsigma_m, \varsigma_{m+2}) = \frac{1}{a} [\kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\vartheta_{m+1}, \vartheta_{m+2})]$, then we have

$$\kappa(\vartheta_m, \vartheta_{m+2}) \le \frac{\gamma}{a}[\kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\vartheta_{m+1}, \vartheta_{m+2})]$$

$$\le \frac{\gamma}{a}\lambda^{m-1}[1 + \lambda^2]\kappa(\vartheta_0, \vartheta_1).$$

Thus

$$\kappa(\vartheta_m, \vartheta_{m+2}) \le \beta_m\, \kappa(\vartheta_0, \vartheta_1), \quad (4)$$

where $\beta_m \in \left\{\lambda^{m-1}, \frac{\gamma(1+\lambda^2)}{1-\gamma}, \gamma\lambda^{m-1}, \gamma\lambda^{m+1}, \frac{\gamma}{a}\lambda^{m-1}[1 + \lambda^2]\right\}, \beta_m \to 0$ as $m \to \infty$.

Now consider $\kappa(\vartheta_m, \vartheta_{m+p})$, we have the following cases:

Case 1. Let p be odd and $p = 2n+1$, where n is a non-negative integer. By rectangular inequality, we have

$$\begin{aligned}\kappa(\vartheta_m,\vartheta_{m+p}) &\le \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+p})\\ &\le \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \kappa(\vartheta_{m+3},\vartheta_{m+4})\\ &\quad + \kappa(\vartheta_{m+4},\vartheta_{m+p})\\ &\le \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \cdots + \kappa(\vartheta_{m+2n-1},\vartheta_{m+2n}) + \kappa(\vartheta_{m+2n},\vartheta_{m+p})\\ &\le \lambda^m\kappa(\vartheta_0,\vartheta_1) + \lambda^{m+1}\kappa(\vartheta_0,\vartheta_1) + \cdots + \lambda^{m+2n}\kappa(\vartheta_0,\vartheta_1) \text{ (using (3))}\\ &\le \left\{\frac{\lambda^m}{1-\lambda}\kappa(\vartheta_0,\vartheta_1)\right\} \downarrow 0.\end{aligned}$$

Case 2. Let p be even and $p = 2n$, where n is a positive integer. By rectangular inequality, we have $\kappa(\vartheta_m,\vartheta_{m+p}) \le$

$$\begin{aligned}&\kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \kappa(\vartheta_{m+3},\vartheta_{m+p})\\ &\le \kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \kappa(\vartheta_{m+3},\vartheta_{m+4}) + \kappa(\vartheta_{m+4},\vartheta_{m+5})\\ &\quad + \kappa(\vartheta_{m+5},\vartheta_{m+p})\\ &\le \kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \cdots + \kappa(\vartheta_{m+2n-1},\vartheta_{m+2n})\\ &\le \beta_m\,\kappa(\vartheta_0,\vartheta_1) + \{\lambda^{m+2} + \lambda^{m+3} + \cdots + \lambda^{m+2n-1}\}\kappa(\vartheta_0,\vartheta_1) \text{ (using (3) and (4))}\\ &\le \left\{\beta_m + \frac{\lambda^{m+2}}{1-\lambda}\kappa(\vartheta_0,\vartheta_1)\right\} \downarrow 0.\end{aligned}$$

Hence $\{\vartheta_m\}$ is $\mathbb{V}$-Cauchy in Z, then there exists a_m in $\mathbb{V}$ such that $a_m \downarrow 0$ and

$$\kappa(\vartheta_m,\vartheta_{m+p}) \preceq a_m. \qquad (5)$$

for all m and p. Since $T(Z) \subseteq S(Z)$ and the range of at least one is $\mathbb{V}$-complete, implies the existence of some $s \in S(Z)$ we have $T\varsigma_m = \vartheta_m = S\varsigma_{m+1} \xrightarrow{\kappa,\mathbb{V}} s$. *So* $\exists$ a sequence $\{r_m\} \in \mathbb{V}$ *s.t.* $r_m \downarrow 0$ and

$$\kappa(\vartheta_m, s) \preceq r_m. \qquad (6)$$

Further since $s \in S(Z)$ then we can find $w \in Z$ s.t. $Sw = s$. Now, we claim that $Tw = s$. We have

$$\begin{aligned}\kappa(Tw,s) &\le \kappa(Tw,T\varsigma_m) + \kappa(T\varsigma_m,T\varsigma_{m+1}) + \kappa(T\varsigma_{m+1},s)\\ &= \kappa(Tw,T\varsigma_m) + \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(T\varsigma_{m+1},s)\\ &\le \gamma H(w,\varsigma_m) + a_m + r_{m+1} \text{ (using (5) and (6))}\\ &\le \gamma H(w,\varsigma_m) + a_m + r_m \ (\because r_{m+1} \le r_m)\end{aligned}$$

where

$$H(w,\varsigma_m) \in \{\kappa(Sw,S\varsigma_m), \kappa(Sw,Tw), \kappa(S\varsigma_m,T\varsigma_m), \tfrac{1}{a}[\kappa(Sw,Tw) + \kappa(S\varsigma_m,T\varsigma_m)]\}$$

$$= \left\{\kappa(s,S\varsigma_m), \kappa(s,Tw), \kappa(\vartheta_{m-1},\vartheta_m), \tfrac{1}{a}[\kappa(s,Tw) + \kappa(\vartheta_{m-1},\vartheta_m)]\right\}.$$

Here, we examine four different cases:

(i) If $H(w,\varsigma_m) = \kappa(s,S\varsigma_m)$, then

$$\begin{aligned}\kappa(Tw,s) &\le \gamma\kappa(s,S\varsigma_m) + a_m + r_m\\ &\le (\gamma+1)r_{m-1} + a_{m-1} \ (\because a_m \le a_{m-1} \text{ and } r_m \le r_{m-1})\end{aligned}$$

implies $\kappa(Tw,s) = 0$.

(ii) If $H(w,\varsigma_m) = \kappa(s,Tw)$, then

$$\kappa(Tw,s) \le \gamma\kappa(s,Tw) + a_m + r_m$$

$$(1-\gamma)\kappa(Tw,s) \le a_m + r_m$$

$$\kappa(Tw,s) \le \frac{1}{1-\gamma}[a_m + r_m]$$

implies $\kappa(Tw,s) = 0$.

(iii) If $H(w,\varsigma_m) = \kappa(\vartheta_{m-1},\vartheta_m)$, then

$$\begin{aligned}\kappa(Tw,s) &\le \gamma\kappa(\vartheta_{m-1},\vartheta_m) + a_m + r_m\\ &\le \gamma a_{m-1} + a_m + r_m \text{ (using (5)}\\ &= (\gamma+1)a_{m-1} + r_m\end{aligned}$$

implies $\kappa(Tw,s) = 0$.

(iv) If $H(w,\varsigma_m) = \frac{1}{a}[\kappa(s,Tw) + \kappa(\vartheta_{m-1},\vartheta_m)]$, then

$$\kappa(Tw,s) \le \frac{\gamma}{a}[\kappa(s,Tw) + \kappa(\vartheta_{m-1},\vartheta_m)] + a_m + r_m$$

$$\frac{a-\gamma}{a}\kappa(Tw,s) \le \frac{\gamma a_{m-1} + a a_m + a r_m}{a}$$

$$\kappa(Tw,s) \le \frac{1}{a-\gamma}[(\gamma+a)a_{m-1} + a r_m]$$

We get $\kappa(s,Tw) = 0$, implies $Tw = s$. Hence s is a PoC of S and T. For proving uniqueness of s, let s_1 be another

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PoC of S and T. Then there is a w_1 in Z s.t. $s_1 = Tw_1 = Sw_1$. Implies

$$\kappa(s, s_1) = \kappa(Tw, Tw_1) \preceq \gamma H(w, w_1)$$

where

$$H(w, w_1) \in \{\kappa(Sw, Sw_1), \kappa(Sw, Tw), \kappa(Sw_1, Tw_1), \frac{1}{a}[\kappa(Sw, Tw) + \kappa(Sw_1, Tw_1)]\}$$
$$= \{\kappa(s, s_1), \kappa(s, s), \kappa(s_1, s_1), \frac{1}{a}[\kappa(s, s) + \kappa(s_1, s_1)]\}$$
$$= \{0, \kappa(s, s_1)\}.$$

This implies $\kappa(s, s_1) = 0$ and so $s = s_1$. Hence S and T have a PoC, which is unique, say z. Further, if both the mappings are WC then by proposition 2.11., z is a unique CFP of S and T.

Theorem 3.2. Let $(Z, \kappa, \mathbb{V})$ be a VVRMS with $\mathbb{V}$-Archimedean and the self mappings S and T on Z satisfies the following conditions:

(i) $\forall\, \varsigma, \vartheta \in Z, \kappa(T\varsigma, T\vartheta) \preceq \gamma H(\varsigma, \vartheta)$

where $\gamma \in [0,1)$, $2 \leq a$ and

$$H(\varsigma, \vartheta) \in \left\{\kappa(S\varsigma, S\vartheta), \frac{1}{a}[\kappa(S\vartheta, T\varsigma) + \kappa(S\varsigma, T\varsigma) + \kappa(S\varsigma, S\vartheta)], \frac{1}{a}[\kappa(S\vartheta, T\varsigma) + \kappa(S\vartheta, T\vartheta) + \kappa(S\varsigma, S\vartheta)]\right\}$$

(ii) $T(Z) \subseteq S(Z)$

(iii) Subspace $T(Z)$ or $S(Z)$, is $\mathbb{V}$ -complete.

Then S and T possess a PoC which is unique in Z. If we assume WC of S and T then there exists a unique CFP of S and T.

Proof. Fix arbitrary $\varsigma_0 \in Z$. Define the sequence $\{\vartheta_m\}$ by $S\varsigma_{m+1} = T\varsigma_m = \vartheta_m$ where $m \geq 0$. Then

$$\kappa(\vartheta_m, \vartheta_{m+1}) = \kappa(T\varsigma_m, T\varsigma_{m+1}) \preceq \gamma H(\varsigma_m, \varsigma_{m+1}),$$

where $H(\varsigma_m, \varsigma_{m+1}) \in \{\kappa(S\varsigma_m, S\varsigma_{m+1}), \frac{1}{a}[\kappa(S\varsigma_{m+1}, T\varsigma_m) + \kappa(S\varsigma_m, T\varsigma_m) + \kappa(S\varsigma_m, S\varsigma_{m+1})],$

$$\frac{1}{a}[\kappa(S\varsigma_{m+1}, T\varsigma_m) + \kappa(S\varsigma_{m+1}, T\varsigma_{m+1}) + \kappa(S\varsigma_m, S\varsigma_{m+1})]\}$$

$$= \{\kappa(\vartheta_{m-1}, \vartheta_m), \frac{1}{a}[\kappa(\vartheta_m, \vartheta_m) + \kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\vartheta_{m-1}, \vartheta_m)]$$
$$\frac{1}{a}[\kappa(\vartheta_m, \vartheta_m) + \kappa(\vartheta_m, \vartheta_{m+1}) + \kappa(\vartheta_{m-1}, \vartheta_m)]\}$$
$$= \{\kappa(\vartheta_{m-1}, \vartheta_m), \frac{2}{a}\kappa(\vartheta_{m-1}, \vartheta_m), \frac{1}{a}[\kappa(\vartheta_m, \vartheta_{m+1}) + \kappa(\vartheta_{m-1}, \vartheta_m)]\}.$$

The possible three cases are:

(i) If $H(\varsigma_m, \varsigma_{m+1}) = \kappa(\vartheta_{m-1}, \vartheta_m)$, then we have

$$\kappa(\vartheta_m, \vartheta_{m+1}) \preceq \gamma\kappa(\vartheta_{m-1}, \vartheta_m).$$

(ii) If $H(\varsigma_m, \varsigma_{m+1}) = \frac{2}{a}\kappa(\vartheta_{m-1}, \vartheta_m)$, we get

$\kappa(\vartheta_m, \vartheta_{m+1}) \preceq \frac{2\gamma}{a}\kappa(\vartheta_{m-1}, \vartheta_m)$ $\left(\text{note that } \frac{2\gamma}{a} < 1\right)$.

(iii) If $H(\varsigma_m, \varsigma_{m+1}) = \frac{1}{a}[\kappa(\vartheta_m, \vartheta_{m+1}) + \kappa(\vartheta_{m-1}, \vartheta_m)]$. Then we have

$$\kappa(\vartheta_m, \vartheta_{m+1}) \preceq \frac{\gamma}{a}[\kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\vartheta_m, \vartheta_{m+1})]$$
$$\left(1 - \frac{\gamma}{a}\right)\kappa(\vartheta_m, \vartheta_{m+1}) \preceq \frac{\gamma}{a}\kappa(\vartheta_{m-1}, \vartheta_m)$$
$$\kappa(\vartheta_m, \vartheta_{m+1}) \preceq \left(\frac{\gamma}{a-\gamma}\right)\kappa(\vartheta_{m-1}, \vartheta_m) \quad \left(\text{note that } \frac{\gamma}{a-\gamma} < 1\right).$$

Thus

$$\kappa(\vartheta_m, \vartheta_{m+1}) \preceq \lambda\,\kappa(\vartheta_{m-1}, \vartheta_m), \tag{7}$$

where $\lambda \in \left\{\gamma, \frac{2\gamma}{a}, \frac{\gamma}{a-\gamma}\right\} < 1$. Repeating the process of (7), we get

$$\kappa(\vartheta_m, \vartheta_{m+1}) \preceq \lambda^m\,\kappa(\vartheta_0, \vartheta_1). \tag{8}$$

Now

$$\kappa(\vartheta_m, \vartheta_{m+2}) = \kappa(T\varsigma_m, T\varsigma_{m+2}) \preceq \gamma H(\varsigma_m, \varsigma_{m+2})$$

where $H(\varsigma_m, \varsigma_{m+2}) \in \{\kappa(S\varsigma_m, S\varsigma_{m+2}), \frac{1}{a}[\kappa(S\varsigma_{m+2}, T\varsigma_m) + \kappa(S\varsigma_m, T\varsigma_m) + \kappa(S\varsigma_m, S\varsigma_{m+2})],$

$$\frac{1}{a}[\kappa(S\varsigma_{m+2}, T\varsigma_m) + \kappa(S\varsigma_{m+2}, T\varsigma_{m+2}) + \kappa(S\varsigma_m, S\varsigma_{m+2})]\}$$
$$= \{\kappa(\vartheta_{m-1}, \vartheta_{m+1}), \frac{1}{a}[\kappa(\vartheta_{m+1}, \vartheta_m) + \kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\vartheta_{m-1}, \vartheta_{m+1})],$$

$$\frac{1}{a}[\kappa(\vartheta_{m+1},\vartheta_m) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+1},\vartheta_{m-1})]\}.$$

The possible cases are:

(i) If $H(\varsigma_m,\varsigma_{m+2}) = \kappa(\vartheta_{m-1},\vartheta_{m+1})$, then we have

$$\kappa(\vartheta_m,\vartheta_{m+2}) \le \gamma\,\kappa(\vartheta_{m-1},\vartheta_{m+1})$$
$$\le \gamma[\kappa(\vartheta_{m-1},\vartheta_m) + \kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+1})]$$
$$(1-\gamma)\kappa(\vartheta_m,\vartheta_{m+2}) \le \gamma[\lambda^{m-1}\kappa(\vartheta_0,\vartheta_1) + \lambda^{m+1}\kappa(\vartheta_0,\vartheta_1)] \text{ (using (8))}$$
$$\kappa(\vartheta_m,\vartheta_{m+2}) \le \lambda^{m-1}\left[\frac{\gamma(1+\lambda^2)}{1-\gamma}\right]\kappa(\vartheta_0,\vartheta_1).$$

(ii) If $H(\varsigma_m,\varsigma_{m+2}) = \frac{1}{a}[\kappa(\vartheta_{m+1},\vartheta_m) + \kappa(\vartheta_{m-1},\vartheta_m) + \kappa(\vartheta_{m-1},\vartheta_{m+1})]$, then we have

$$\kappa(\vartheta_m,\vartheta_{m+2}) \le \frac{\gamma}{a}[\kappa(\vartheta_{m+1},\vartheta_m) + \kappa(\vartheta_{m-1},\vartheta_m) + \kappa(\vartheta_{m-1},\vartheta_{m+1})]$$
$$\le \frac{\gamma}{a}[\kappa(\vartheta_{m+1},\vartheta_m) + \kappa(\vartheta_{m-1},\vartheta_m) + \kappa(\vartheta_{m-1},\vartheta_m) + \kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+1})]$$
$$\left(1-\frac{\gamma}{a}\right)\kappa(\vartheta_m,\vartheta_{m+2}) \le \frac{\gamma}{a}[\lambda^m + \lambda^{m-1} + \lambda^{m-1} + \lambda^{m+1}]\kappa(\vartheta_0,\vartheta_1) \text{ (using (8))}$$
$$\kappa(\vartheta_m,\vartheta_{m+2}) \le \frac{\gamma\lambda^{m-1}}{a-\gamma}[\lambda(1+\lambda) + 2]\kappa(\vartheta_0,\vartheta_1).$$

(iii) If If $H(\varsigma_m,\varsigma_{m+2}) = \frac{1}{a}[\kappa(\vartheta_{m+1},\vartheta_m) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+1},\vartheta_{m-1})]$, then we have

$$\kappa(\vartheta_m,\vartheta_{m+2}) \le \frac{\gamma}{a}[\kappa(\vartheta_{m+1},\vartheta_m) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+1},\vartheta_{m-1})]$$
$$\le \frac{\gamma}{a}[\kappa(\vartheta_{m+1},\vartheta_m) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_m)] + \kappa(\vartheta_m,\vartheta_{m-1})]$$
$$\left(1-\frac{\gamma}{a}\right)\kappa(\vartheta_m,\vartheta_{m+2}) \le \frac{\gamma}{a}[\lambda^m + \lambda^{m+1} + \lambda^{m+1} + \lambda^{m-1}]\kappa(\vartheta_0,\vartheta_1) \text{ (using (8))}$$
$$\kappa(\vartheta_m,\vartheta_{m+2}) \le \frac{\gamma\lambda^{m-1}}{a-\gamma}[\lambda(1+2\lambda) + 1]\kappa(\vartheta_0,\vartheta_1).$$

Thus

$$\kappa(\vartheta_m,\vartheta_{m+2}) \le \beta_m\,\kappa(\vartheta_0,\vartheta_1). \quad (9)$$

where $\beta_m \in \left\{\lambda^{m-1}\left[\frac{\gamma(1+\lambda^2)}{1-\gamma}\right], \frac{\gamma\lambda^{m-1}}{a-\gamma}[\lambda(1+\lambda)+2], \frac{\gamma\lambda^{m-1}}{a-\gamma}[\lambda(1+2\lambda)+1]\right\}$, $\beta_m \to 0$ as $m \to \infty$.

Now consider $\kappa(\vartheta_m,\vartheta_{m+p})$, we have the following cases:

Case 1. Let p be odd and $p = 2n+1$, where n is a non-negative integer. By rectangular inequality, we have

$$\kappa(\vartheta_m,\vartheta_{m+p}) \le \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+p})$$
$$\le \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \kappa(\vartheta_{m+3},\vartheta_{m+4}) + \kappa(\vartheta_{m+4},\vartheta_{m+p})$$
$$\le \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \cdots + \kappa(\vartheta_{m+2n-1},\vartheta_{m+2n}) + \kappa(\vartheta_{m+2n},\vartheta_{m+p})$$
$$\le \lambda^m\kappa(\vartheta_0,\vartheta_1) + \lambda^{m+1}\kappa(\vartheta_0,\vartheta_1) + \cdots + \lambda^{m+2n}\kappa(\vartheta_0,\vartheta_1) \text{ (using (8))}$$
$$\le \left\{\frac{\lambda^m}{1-\lambda}\kappa(\vartheta_0,\vartheta_1)\right\} \downarrow 0.$$

Case 2. Let p be even and $p = 2n$, where n is a positive integer. By rectangular inequality, we have $\kappa(\vartheta_m,\vartheta_{m+p}) \le$

$$\kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \kappa(\vartheta_{m+3},\vartheta_{m+p})$$
$$\le \kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \kappa(\vartheta_{m+3},\vartheta_{m+4}) + \kappa(\vartheta_{m+4},\vartheta_{m+5}) + \kappa(\vartheta_{m+5},\vartheta_{m+p})$$
$$\le \kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \cdots + \kappa(\vartheta_{m+2n-1},\vartheta_{m+2n})$$
$$\le \beta_m\,\kappa(\vartheta_0,\vartheta_1) + \{\lambda^{m+2} + \lambda^{m+3} + \cdots + \lambda^{m+2n-1}\}\kappa(\vartheta_0,\vartheta_1) \text{ (using (8) and (9))}$$
$$\le \left\{\beta_m + \frac{\lambda^{m+2}}{1-\lambda}\kappa(\vartheta_0,\vartheta_1)\right\} \downarrow 0.$$

Hence $\{\vartheta_m\}$ is $\mathbb{V}$-Cauchy in Z, so then there exists a_m in $\mathbb{V}$ such that $a_m \downarrow 0$ and

$$\kappa(\vartheta_m,\vartheta_{m+p}) \le a_m. \quad (10)$$

for all m and p. Since T(Z) ⊆ S(Z) and the range of at least one is $\mathbb{V}$-complete, implies the existence of some $s \in S(Z)$ we have $T\varsigma_m = \vartheta_m = S\varsigma_{m+1} \xrightarrow{\kappa,\mathbb{V}} s.$ $So \ \exists\ \{r_m\} \in \mathbb{V}$ s.t. $r_m \downarrow 0$ and

$$\kappa(\vartheta_m,s) \le r_m \quad (11)$$

Further since $s \in S(Z)$ then we can find $w \in Z$ s.t. $Sw = s$. Now, we claim that $Tw = s$. We have

$$\kappa(Tw,s) \le \kappa(Tw,T\varsigma_m) + \kappa(T\varsigma_m,T\varsigma_{m+1}) + \kappa(T\varsigma_{m+1},s)$$
$$= \kappa(Tw,T\varsigma_m) + \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(T\varsigma_{m+1},s)$$

$$\leq \gamma H(w,\varsigma_m) + a_m + r_{m+1} \text{ (using (10) and (11)}$$
$$\leq \gamma H(w,\varsigma_m) + a_m + r_m \ (\because r_{m+1} \leq r_m),$$

where $H(w,\varsigma_m) \in \{\kappa(Sw,S\varsigma_m), \frac{1}{a}[\kappa(S\varsigma_m,Tw) + \kappa(Sw,Tw) + \kappa(Sw,S\varsigma_m)],$

$$\frac{1}{a}[\kappa(S\varsigma_m,Tw) + \kappa(S\varsigma_m,T\varsigma_m) + \kappa(Sw,S\varsigma_m)]\}$$
$$= \{\kappa(s,S\varsigma_m), \frac{1}{a}[\kappa(S\varsigma_m,Tw) + \kappa(s,Tw) + \kappa(s,S\varsigma_m)],$$
$$\frac{1}{a}[\kappa(S\varsigma_m,Tw) + \kappa(S\varsigma_m,T\varsigma_m) + \kappa(s,S\varsigma_m)]\}.$$

Here, we examine three different cases:

(i) If $H(w,\varsigma_m) = \kappa(s,S\varsigma_m)$, then

$$\kappa(Tw,s) \leq \gamma\kappa(s,S\varsigma_m) + a_m + r_m$$
$$\leq (\gamma + 1)r_{m-1} + r_{m-1} \ (\because a_m \leq a_{m-1} \text{ and } r_m \leq r_{m-1}).$$

implies $\kappa(Tw,s) = 0$.

(ii) If $H(w,\varsigma_m) = \frac{1}{a}[\kappa(S\varsigma_m,Tw) + \kappa(s,Tw) + \kappa(s,S\varsigma_m)]$, then

$$\kappa(Tw,s) \leq \frac{\gamma}{a}[\kappa(S\varsigma_m,Tw) + \kappa(s,Tw) + \kappa(s,S\varsigma_m)] + a_m + r_m$$
$$\leq \frac{\gamma}{a}[\kappa(S\varsigma_m,T\varsigma_m) + \kappa(T\varsigma_m,s) + \kappa(s,Tw) + \kappa(s,Tw) + \kappa(s,S\varsigma_m)] + a_m + r_m$$
$$\leq \frac{\gamma}{a}[\kappa(\vartheta_{m-1},\vartheta_m) + \kappa(T\varsigma_m,s) + \kappa(s,Tw) + \kappa(s,Tw) + \kappa(s,S\varsigma_m)] + a_m + r_m$$
$$\left(1 - \frac{2\gamma}{a}\right)\kappa(Tw,s) \leq \frac{\gamma}{a}[a_{m-1} + 2b_{m-1}] + a_m + r_m \ (\because \mathrm{r_m} \leq \mathrm{r_{m-1}})$$
$$\kappa(Tw,s) \leq \left(\frac{\gamma}{a-2\gamma} + 1\right)a_{m-1} + \left(\frac{2\gamma}{a-2\gamma} + 1\right)r_{m-1}$$
$$= \frac{a-\gamma}{a-2\gamma}a_{m-1} + \frac{a}{a-2\gamma}r_{m-1},$$

implies $\kappa(Tw,s) = 0$.

(iii) If $H(w,\varsigma_m) = \frac{1}{a}[\kappa(S\varsigma_m,Tw) + \kappa(S\varsigma_m,T\varsigma_m) + \kappa(s,S\varsigma_m)]$, then

$$\kappa(Tw,s) \leq \frac{\gamma}{a}[\kappa(S\varsigma_m,Tw) + \kappa(S\varsigma_m,T\varsigma_m) + \kappa(s,S\varsigma_m)] + a_m + r_m$$
$$\leq \frac{\gamma}{a}[\kappa(S\varsigma_m,T\varsigma_m) + \kappa(T\varsigma_m,s) + \kappa(s,Tw) + \mathrm{a_{m-1}} + \mathrm{r_{m-1}}] + \mathrm{a_m} + \mathrm{r_m}$$
$$\left(1 - \frac{\gamma}{a}\right)\kappa(Tw,s) \leq \frac{\gamma}{a}[a_{m-1} + r_m + a_{m-1} + r_{m-1}] + a_{m-1} + r_{m-1}$$
$$\kappa(Tw,s) \leq \left(\frac{2\gamma}{a-\gamma} + 1\right)(a_{m-1} + r_{m-1})$$
$$= \frac{a+\gamma}{a-\gamma}(a_{m-1} + r_{m-1}).$$

We get $\kappa(s,Tw) = 0$, implies $Tw = s$. Hence s is a PoC of S and T. For proving uniqueness of s, let s_1 be another PoC of S and T. Then there is a w_1 in Z s.t. $s_1 = Tw_1 = Sw_1$. Implies

$$\kappa(s,s_1) = \kappa(Tw,Tw_1) \leq \gamma H(w,w_1)$$

where $H(w,w_1) \in \{\kappa(Sw,Sw_1), \frac{1}{a}[\kappa(Sw_1,Tw) + \kappa(Sw,Tw) + \kappa(Sw,Sw_1)],$

$$\frac{1}{a}[\kappa(Sw_1,Tw) + \kappa(Sw_1,Tw_1) + \kappa(Sw,Sw_1)]\}$$
$$= \{\kappa(s,s_1), \frac{1}{a}[\kappa(s_1,s) + \kappa(s,s) + \kappa(s,s_1)],$$
$$\frac{1}{a}[\kappa(s_1,s) + \kappa(s_1,s_1) + \kappa(s,s_1)]\}$$
$$= \left\{\kappa(s,s_1), \frac{2}{a}\kappa(s_1,s)\right\}.$$

The following two cases arise:

(i) If $H(w,w_1) = \kappa(s,s_1)$, then

$$\kappa(s,s_1) \leq \gamma\kappa(s,s_1).$$

Thus $\kappa(s,s_1) = 0$, implies $s = s_1$.

(ii) If $H(w,w_1) = \frac{2}{a}\kappa(s_1,s)$, then

$\kappa(s,s_1) \preceq \frac{2\gamma}{a}\kappa(s_1,s)$.

Since $\frac{2\gamma}{a} < 1$, we get $\kappa(s,s_1) = 0$ implies $s = s_1$. Hence S and T have a PoC, which is unique, say z. Further, if both the mappings are WC then by proposition 2.11., z is a unique CFP of S and T.

Theorem 3.3. Let $(Z,\kappa,\mathbb{V})$ be a VVRMS with $\mathbb{V}$-Archimedean and the self mappings S and T on Z satisfies the following conditions:

(i) $\forall\, \varsigma,\vartheta \in Z,\ \kappa(T\varsigma,T\vartheta) \preceq \alpha_1\kappa(S\varsigma,T\varsigma) + \alpha_2\kappa(S\vartheta,T\vartheta) + \alpha_3\kappa(S\varsigma,S\vartheta)$
where $0 < \alpha_1,\alpha_2,\alpha_3$ and $\sum_{i=1}^{3}\alpha_i < 1$

(ii) $T(Z) \subseteq S(Z)$

(iii) Subspace $T(Z)$ or $S(Z)$, is $\mathbb{V}$ -complete.

Then S and T possess a PoC which is unique in Z. If we assume WC of S and T then there exists a unique CFP of S and T.

Proof. Fix arbitrary $\varsigma_0 \in Z$. Define the sequence $\{\vartheta_m\}$ by $S\varsigma_{m+1} = T\varsigma_m = \vartheta_m$ where $m \geq 0$. Then

$$\begin{aligned}\kappa(\vartheta_m,\vartheta_{m+1}) = \kappa(T\varsigma_m,T\varsigma_{m+1}) &\preceq \alpha_1\kappa(S\varsigma_m,T\varsigma_m) + \alpha_2\kappa(S\varsigma_{m+1},T\varsigma_{m+1}) + \alpha_3\kappa(S\varsigma_m,S\varsigma_{m+1})\\ &= \alpha_1\kappa(\vartheta_{m-1},\vartheta_m) + \alpha_2\kappa(\vartheta_m,\vartheta_{m+1}) + \alpha_3\kappa(\vartheta_{m-1},\vartheta_m)\\ &= (\alpha_1+\alpha_3)\kappa(\vartheta_{m-1},\vartheta_m) + \alpha_2\kappa(\vartheta_m,\vartheta_{m+1}).\end{aligned}$$

And

$$\begin{aligned}\kappa(\vartheta_{m+1},\vartheta_m) = \kappa(T\varsigma_{m+1},T\varsigma_m) &\preceq \alpha_1\kappa(S\varsigma_{m+1},T\varsigma_{m+1}) + \alpha_2\kappa(S\varsigma_m,T\varsigma_m) + \alpha_3\kappa(S\varsigma_{m+1},S\varsigma_m)\\ &= \alpha_1\kappa(\vartheta_m,\vartheta_{m+1}) + \alpha_2\kappa(\vartheta_{m-1},\vartheta_m) + \alpha_3\kappa(\vartheta_m,\vartheta_{m-1})\\ &= \alpha_1\kappa(\vartheta_m,\vartheta_{m+1}) + (\alpha_2+\alpha_3)\kappa(\vartheta_{m-1},\vartheta_m).\end{aligned}$$

Hence

$$2\kappa(\vartheta_m,\vartheta_{m+1}) \preceq (\alpha_1+\alpha_2)\kappa(\vartheta_m,\vartheta_{m+1}) + (\alpha_1+\alpha_2+2\alpha_3)\kappa(\vartheta_{m-1},\vartheta_m)$$

$$\begin{aligned}\kappa(\vartheta_m,\vartheta_{m+1}) &\preceq \frac{\alpha_1+\alpha_2+2\alpha_3}{2-\alpha_1-\alpha_2}\kappa(\vartheta_{m-1},\vartheta_m)\\ &= \lambda\kappa(\vartheta_{m-1},\vartheta_m). \qquad (12)\end{aligned}$$

where $\lambda = \frac{\alpha_1+\alpha_2+2\alpha_3}{2-\alpha_1-\alpha_2} < 1$. Repeating the process of (12), we get

$$\kappa(\vartheta_m,\vartheta_{m+1}) \preceq \lambda^m\,\kappa(\vartheta_0,\vartheta_1). \qquad (13)$$

Now

$$\begin{aligned}\kappa(\vartheta_m,\vartheta_{m+2}) &= \kappa(T\varsigma_m,T\varsigma_{m+2})\\ &\preceq \alpha_1\kappa(S\varsigma_m,T\varsigma_m) + \alpha_2\kappa(S\varsigma_{m+2},T\varsigma_{m+2}) + \alpha_3\kappa(S\varsigma_m,S\varsigma_{m+2})\\ &= \alpha_1\kappa(\vartheta_{m-1},\vartheta_m) + \alpha_2\kappa(\vartheta_{m+1},\vartheta_{m+2}) + \alpha_3\kappa(\vartheta_{m-1},\vartheta_{m+1})\\ &\preceq \alpha_1\lambda^{m-1}\kappa(\vartheta_0,\vartheta_1) + \alpha_2\lambda^{m+1}\kappa(\vartheta_0,\vartheta_1) + \alpha_3[\kappa(\vartheta_{m-1},\vartheta_m)\\ &\quad +\kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+1})] \text{ (using (13))}\end{aligned}$$

$$(1-\alpha_3)\kappa(\vartheta_m,\vartheta_{m+2}) \preceq [\alpha_1\lambda^{m-1} + \alpha_2\lambda^{m+1}]\kappa(\vartheta_0,\vartheta_1) + \alpha_3[\lambda^{m-1} + \lambda^{m+1}]\,\kappa(\vartheta_0,\vartheta_1)$$

$$\kappa(\vartheta_m,\vartheta_{m+2}) \preceq \left(\frac{\alpha_1+\alpha_3}{1-\alpha_3}\lambda^{m-1} + \frac{\alpha_2+\alpha_3}{1-\alpha_3}\lambda^{m+1}\right)\kappa(\vartheta_0,\vartheta_1).$$

Thus

$$\kappa(\vartheta_m,\vartheta_{m+2}) \preceq \beta_m\,\kappa(\vartheta_0,\vartheta_1), \qquad (14)$$

where $\beta_m = \frac{\alpha_1+\alpha_3}{1-\alpha_3}\lambda^{m-1} + \frac{\alpha_2+\alpha_3}{1-\alpha_3}\lambda^{m+1}$, $\beta_m \to 0$ as $m \to \infty$.

Now consider $\kappa(\vartheta_m,\vartheta_{m+p})$, we have the following cases:

Case 1. Let p be odd and $p = 2n+1$, where n is a non-negative integer. By rectangular inequality, we have

$$\begin{aligned}\kappa(\vartheta_m,\vartheta_{m+p}) &\preceq \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+p})\\ &\preceq \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \kappa(\vartheta_{m+3},\vartheta_{m+4})\\ &\quad + \kappa(\vartheta_{m+4},\vartheta_{m+p})\\ &\preceq \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \cdots + \kappa(\vartheta_{m+2n-1},\vartheta_{m+2n}) + \kappa(\vartheta_{m+2n},\vartheta_{m+p})\\ &\preceq \lambda^m\kappa(\vartheta_0,\vartheta_1) + \lambda^{m+1}\kappa(\vartheta_0,\vartheta_1) + \cdots + \lambda^{m+2n}\kappa(\vartheta_0,\vartheta_1) \text{ (using (13))}\\ &\preceq \left\{\frac{\lambda^m}{1-\lambda}\kappa(\vartheta_0,\vartheta_1)\right\} \downarrow 0.\end{aligned}$$

Case 2. Let p be even and $p = 2n$, where n is a positive integer. By rectangular inequality, we have

$$\begin{aligned}\kappa(\vartheta_m,\vartheta_{m+p}) &\preceq \kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \kappa(\vartheta_{m+3},\vartheta_{m+p})\\ &\preceq \kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \kappa(\vartheta_{m+3},\vartheta_{m+4}) + \kappa(\vartheta_{m+4},\vartheta_{m+5})\\ &\quad + \kappa(\vartheta_{m+5},\vartheta_{m+p})\\ &\preceq \kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \cdots + \kappa(\vartheta_{m+2n-1},\vartheta_{m+2n})\\ &\preceq \beta_m\,\kappa(\vartheta_0,\vartheta_1) + \{\lambda^{m+2} + \lambda^{m+3} + \cdots + \lambda^{m+2n-1}\}\kappa(\vartheta_0,\vartheta_1) \text{ (using (13) and (14))}\end{aligned}$$

$$\leq \left\{\beta_m + \frac{\lambda^{m+2}}{1-\lambda}\kappa(\vartheta_0,\vartheta_1)\right\} \downarrow 0.$$

Hence $\{\vartheta_m\}$ is $\mathbb{V}$-Cauchy in Z, so then there exists a_m in $\mathbb{V}$ such that $a_m \downarrow 0$ and

$$\kappa(\vartheta_m,\vartheta_{m+p}) \leq a_m, \qquad (15)$$

for all m and p. Since T(Z) $\subseteq$ S(Z) and the range of at least one is $\mathbb{V}$-complete, implies the existence of some $s \in S(Z)$ we have $T\varsigma_m = \vartheta_m = S\varsigma_{m+1} \overset{\kappa,\mathbb{V}}{\longrightarrow} s.$ So $\exists\ \{r_m\} \in \mathbb{V}$ s.t. $r_m \downarrow 0$ and

$$\kappa(\vartheta_m, s) \leq r_m. \qquad (16)$$

Further since $s \in S(Z)$ then we can find $w \in Z$ s.t. $Sw = s$. Now, we claim that $Tw = s$. We have

$$\kappa(Tw,s) \leq \kappa(Tw,T\varsigma_m) + \kappa(T\varsigma_m,T\varsigma_{m+1}) + \kappa(T\varsigma_{m+1},s)$$
$$\leq \alpha_1\kappa(Sw,Tw) + \alpha_2\kappa(S\varsigma_m,T\varsigma_m) + \alpha_3\kappa(Sw,S\varsigma_m) + \kappa(\vartheta_m,\vartheta_{m+1}) + r_{m+1}$$
$$\leq \alpha_1\kappa(s,Tw) + \alpha_2\kappa(\vartheta_{m-1},\vartheta_m) + \alpha_3\kappa(s,S\varsigma_m) + a_m + r_m \ (\because\ r_{m+1} \leq r_m)$$
$$(1-\alpha_1)\,\kappa(Tw,s) \leq \alpha_2 a_{m-1} + \alpha_3 r_{m-1} + a_{m-1} + r_{m-1}$$
$$\kappa(Tw,s) \leq \frac{1+\alpha_2}{1-\alpha_1}a_{m-1} + \frac{1+\alpha_2}{1-\alpha_1}r_{m-1}.$$

We get $\kappa(Tw,s) = 0$, implies $Tw = s$. Hence s is a PoC of S and T. For proving uniqueness of s, let s_1 be another PoC of S and T. Then there is a w_1 in Z s.t. $s_1 = Tw_1 = Sw_1$. Implies

$$\kappa(s,s_1) = \kappa(Tw,Tw_1) \leq \alpha_1\kappa(Sw,Tw) + \alpha_2\kappa(Sw_1,Tw_1) + \alpha_3\kappa(Sw,Sw_1)$$
$$= \alpha_1\kappa(s,s) + \alpha_2\kappa(s_1,s_1) + \alpha_3\kappa(s,s_1)$$
$$= \alpha_3\kappa(s,s_1).$$

Since $\alpha_3 < 1$, implies $s = s_1$. Hence S and T have a PoC, which is unique, say z. Further, if both the mappings are WC then by proposition 2.11., z is a unique CFP of S and T.

Corollary 3.4. Let $(Z,\kappa,\mathbb{V})$ be a VVRMS with $\mathbb{V}$-Archimedean and the self mappings S and T on Z satisfies the following conditions:

(i) $\forall\ \varsigma,\vartheta \in Z,\ \kappa(T\varsigma,T\vartheta) \leq \gamma\kappa(S\varsigma,S\vartheta)$
where $\gamma \in [0,1)$
(ii) $T(Z) \subseteq S(Z)$
(iii) Subspace $T(Z)$ or $S(Z)$, is $\mathbb{V}$ -complete.

Then S and T possess a PoC which is unique in Z. If we assume WC of S and T then there exists a unique CFP of S and T.

Proof. Result is obtained by choosing $\alpha_1 = 0 = \alpha_2, \alpha_3 = \gamma$ in Theorem 3.3.

Corollary 3.5. Let $(Z,\kappa,\mathbb{V})$ be a VVRMS with $\mathbb{V}$-Archimedean and the self mappings S and T on Z satisfies the following conditions:

(i) $\forall\ \varsigma,\vartheta \in Z,\ \kappa(T\varsigma,T\vartheta) \leq \gamma[\kappa(S\varsigma,T\varsigma) + \kappa(S\vartheta,T\vartheta)]$
where $\gamma \in [0,\frac{1}{2})$
(ii) $T(Z) \subseteq S(Z)$
(iii) Subspace $T(Z)$ or $S(Z)$, is $\mathbb{V}$ -complete.

Then S and T possess a PoC which is unique in Z. If we assume WC of S and T then there exists a unique CFP of S and T.

Proof. Result is obtained by choosing $\alpha_1 = 0 = \alpha_2, \alpha_3 = \gamma$ in Theorem 3.3.

Example 3.6. Let $Z = R, \mathbb{V} = R^2$ with coordinatewise ordering and for all $\varsigma,\vartheta \in Z$ mapping $\kappa : Z \times Z \to \mathbb{V}$ such that

$\kappa(\varsigma,\vartheta) = (|\varsigma - \vartheta|, \alpha|\varsigma - \vartheta|)$,

where $\alpha > 0$ is a constant. Then space $(Z,\kappa,\mathbb{V})$ is VVRMS.

Define self mappings T and S on set Z such that

$$T(\varsigma) = \begin{cases} \dfrac{\alpha\varsigma}{\beta+1} : \text{if } \varsigma \neq 0 \\ \gamma \quad : \text{if } \varsigma = 0 \end{cases}$$

and

$$S(\varsigma) = \begin{cases} a\varsigma : \text{ if } \varsigma \neq 0 \\ \gamma : \text{if } \varsigma = 0 \end{cases}$$

where $\beta \geq 1$, and $\gamma \neq 0$. If $(\varsigma,\vartheta) = (0,0)$, then $\kappa(T(\varsigma),T(\vartheta)) = \kappa(\gamma,\gamma) = 0$, and if $(\varsigma,\vartheta) \neq (0,0)$, then we have

$$\kappa(\mathrm{T}(\varsigma), \mathrm{T}(\vartheta)) = \kappa\left(\frac{\alpha\varsigma}{\beta+1}, \frac{\alpha\vartheta}{\beta+1}\right)$$
$$\preceq \frac{1}{\beta+1}(|\,\alpha\varsigma - \alpha\vartheta\,|, \alpha\,|\,\alpha\varsigma - \alpha\vartheta\,|)$$
$$\preceq \frac{1}{\beta}\kappa(S(\varsigma), S(\vartheta)).$$

Thus we have $\kappa(T(\varsigma), T(\vartheta)) \preceq \lambda\kappa(S(\varsigma), S(\vartheta)), \forall \varsigma, \vartheta \in Z$, where $\lambda = \frac{1}{\beta} \in (0,1]$. Since $S(0) = T(0)$ then 0 is a coincidence point of T and S in W. Here mappings T and S are not WC since

$$\mathrm{T\,S}(0) = \frac{\alpha\varsigma}{\beta+1} \neq \alpha\gamma = \mathrm{ST}(0).$$

Further T and S do not have CFP.

Example 3.7. Let $Z = R, \mathbb{V} = R^2$ with coordinatewise ordering and for all $\varsigma, \vartheta \in Z$ mapping $\kappa : Z \times Z \to \mathbb{V}$ such that

$$\kappa(\varsigma, \vartheta) = (|\,\varsigma - \vartheta\,|, \alpha\,|\,\varsigma - \vartheta\,|)$$

where $\alpha > 0$ is a constant. Then space $(Z, \kappa, \mathbb{V})$ is VVRMS.
Define self mappings T and S on set Z such that

$T\varsigma = \varsigma^2$ and $S\varsigma = 2\varsigma^2 - 1$.

Then $\forall\, \varsigma, \vartheta \in Z$, we observe that
(i) $\kappa(T\varsigma, T\,\vartheta) = \frac{1}{2}\kappa(S\varsigma, S\vartheta) \preceq \lambda\kappa(S\varsigma, S\vartheta)$ where $\lambda \in \left[\frac{1}{2}, 1\right)$
(ii) $T(Z) = [0, \infty) \subseteq [-1, \infty) = S(Z)$ and $T(Z)$ is $\mathbb{V}$ -complete subspace of Z
Then $1, -1$ are coincidence points of T and S and 1 is unique PoC. Also mappings T and S are WC since

$T(S(1)) = T(1) = 1 = S(1) = S(T(1))$ and $T(S(-1)) = T(1) = 1 = S(1) = S(T(-1))$.

Then 1 is a unique CFP of T and S.

Conclusion

Motivated by the common fixed point results available in literature for metric space [10, 11, 14], rectangular metric space [6, 8] and Riesz space valued metric spaces [3, 7, 12, 13]. We carried on our study of common fixed point result on vector valued rectangular metric spaces. In this case, the metric is Riesz valued. Our results generalize some of the known results. However, there is enough scope to investigate these problems when instead of Riesz space if we take some other space.

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