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The Vertex Strong Geodetic Number of a Graph

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Abstract

Let x be a vertex of G and $S \subseteq V - \{x\}$. Then for each vertex $y \in S$, $x \neq y$. Let $\tilde{g}_x[y]$ be a selected fixed shortest x-y path. Then we set $\tilde{I}_x[S] = \{\tilde{g}_x(y): y \in S\}$ and let $V(\tilde{I}_x[S]) = \bigcup_{p \in \tilde{I}_x[S]} V(P)$. If $V(\tilde{I}_x[S]) = V$ for some $\tilde{I}_x[S]$ then the set S is called a vertex strong geodetic set of G. The minimum cardinality of a vertex strong geodetic set of G is called the vertex strong geodetic number of G and is denoted by $sg_x(G)$. Some of the standard graphs are determined. Necessary conditions for $sg_x(G)$ to be $n-1$ is given for some vertex x in G. It is shown for every pair of integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $sg(G) = b + 2$ and $sg_x(G) = a + b + 1$ for some x in G.

Keywords: strong geodetic number, vertex strong geodetic number, geodetic number.

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [1]. Two vertices u and v are said to be *adjacent* if uv is an edge of G. Two edges of G are said to be adjacent if they have a common vertex. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G.

An $u{-}v$ path of length $d(u, v)$ is called an $u{-}v$ *geodesic*. An $x - y$ path of length $d(x, y)$ is called geodesic. A vertex v is said to lie on a geodesic P if v is an internal vertex of P. The closed interval $I[x, y]$ consists of x, y and all vertices lying on some $x - y$ geodesic of G and for a non-empty set $S \subseteq V(G)$, $I[S] = \cup_{x,y \in S} I[x, y]$.

A set $S \subseteq V(G)$ in a connected graph G is a geodetic set of G if $I[S] = V(G)$. The geodetic number of G, denoted by $g(G)$, is the minimum cardinality of a geodetic set of G. The geodetic concept were studied in [1, 3, 4]. Let $S \subset V(G)$ and $x \in V$ such that $x \notin S$. Let $I_x[y]$ be the set of all vertices that lies in x-y geodesic including x and y, where `$y \in S$ and $I_x[S] = \bigcup_{y \in S} I_x[y]$. Then S is said to be an x-geodetic set of G, if $I_x[S] = V$. The x-geodetic concept were studied in [10]. The following theorem is used in sequel.

Theorem 1.1 [10] Every extreme vertex of G other than the vertex x (whether x is extreme or not) belongs to every x-geodetic set for any vertex x in G

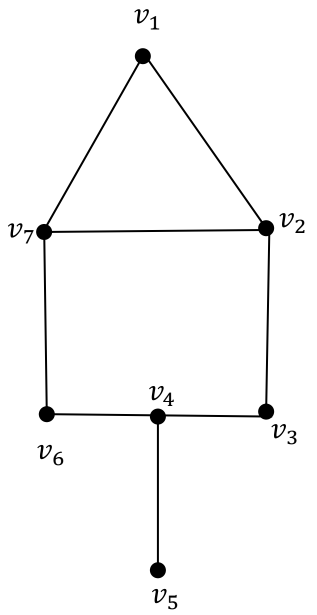
2.The Vertex Strong Geodetic Number of a Graph

Definition 2.1. Let x be a vertex of G and $S \subseteq V - \{x\}$. Then for each vertex y $\in S$, $x \neq y$.

Let $\tilde{g}_x[y]$ be a selected fixed shortest x-y path. Then we set $\tilde{I}_x[S] = \{\tilde{g}_x(y): y \in S\}$ and let $V(\tilde{I}_x[S]) = \bigcup_{p \in \tilde{I}_x[S]} V(P)$.

If $V(\tilde{I}_x[S]) = V$ for some $\tilde{I}_x[S]$ then the set S is called a vertex strong geodetic set of G. The minimum cardinality of a vertex strong geodetic set of G is called the vertex strong geodetic number of G and is denoted by $sg_x(G)$.

Example 2.2. For the graph G given in Figure 2.1, sg_x-sets and $sg_x(G)$ for each vertex x is given in the following Table 2.1.



G

Figure 2.1

Table 2.1

Vertex	sg_x-sets	$sg_x(G)$
v_1	$\{v_5, v_6\}$	2
v_2	$\{v_1, v_5, v_6\}$	3
v_3	$\{v_1, v_5, v_7\}$	3
v_4	$\{v_1, v_5, v_7\}$	3
v_5	$\{v_1, v_7\}$	2
v_6	$\{v_1, v_5, v_7\}$	3

Note 2.3. Every vertex of an x-y geodesic in x- vertex strong geodetic the vertex y. Since by definition a sg_x-sets is minimum, the vertex x and also the internal vertices of an x-y geodesic do not belong to a sg_x-set.

Theorem 2.4. For any vertex x in G, sg_x-set is unique and it is contained in every x- vertex strong geodetic set of G.
Proof. Suppose there are two sg_x-sets say S_1 and S_2. Let u be a vertex of G such that $u \in S_1$ and $u \notin S_2$. Since S_2 is a sg_x-set, $|S_2| = |S_1|$ and hence there exists a vertex $v \neq u$ in G such that $v \in S_2$ and $v \notin S_1$. Since S_1 is a sg_x-set and $v \notin S_1$, there exists a vertex $w \in S_1$, such that $v \in I[x, w]$.
Case 1. Suppose $w \in S_2$. Since v is an internal vertex of an x-w geodesic and S_2 is a sg_x-set, v is not in S_2, which is a contradiction to $v \in S_2$ ________(1)
Case 2. Suppose $w \notin S_2$. Since S_2 is a sg_x-set, there exists an element $y \in S_2$ such that w lies an x-y geodesic say P. From (1), v lies on an x-w geodesic say Q. Then the union of the geodesic Q from x to w and the w-y section of the geodesic P is an x-y geodesic so that $v \in I[x, y]$. Thus v is an internal vertex of an x-y geodesic. Since S_2 is a sg_x-set, v is not in S_2, Which is a contradiction to $v \notin S_2$.

Now claim that sg_x-set is contained in every vertex strong geodetic set of x of G. Let y be an element of the sg_x-set. say S of G. Since S is minimum, $y \notin I_x[z]$ for any other vertex z in G. If there exists vertex strong geodetic set of x of G. say S', such that $y \notin S'$, then y lies on an x-v geodesic for some $v \in S'$ and hence $y \in I_x[v]$, which is a contradiction.
Observation 2.5. Let G be a connected graph

(i) Every simplicial vertex of G other than the vertex x (whether x is simplicial or not) belongs to the sg_x-set for any vertex x in G.
(ii) For any vertex x, eccentric vertices of x belong to the sg_x-set.
(iii) No cut vertex of G belongs to any sg_x-set.

Note 2.6. Even if x is a simplicial vertex of G, x does not belong to the sg_x-set.

Corollary 2.7. Let T be a tree with number of end vertices k. Then $sg_x(T) = k - 1$ or k according as x is an end or non-end vertex of T.

Proof. This follows from Observation 2.4. ■

Corollary 2.8. Let P_n be a non-trivial path. Then $sg_x(P_n) = 1$ or 2 according as x is an end or non-end vertex.

Theorem 2.9. For the cycle $G = C_n (n \geq 4)$, then $sg_x(C_n) = 2$ for every $x \in G$.

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Let x be a vertex of G.
Let n be even. Let y be the antipodal vertex of x. Then $\{y\}$ is not a vertex strong geodetic set of G. Fix the $x - y$ geodesic P. Let P_1 be another x-y geodesic in G. Let z be a vertex in P_1 such that $yz \in V(C_n)$. Let $S = \{y, z\}$. Then S is a vertex strong geodetic set of G so that $sg_x(C_n) = 2$.

Next assume that n is odd. It is easily verified that $sg_x(C_n) \geq 2$. Let y and z be the two antipodal vertices of x. Then $S_1 = \{y, z\}$ is a vertex strong geodetic set of G so that $sg_x(C_n) = 2$. ■

Corollary 2.10. (i) Let $K_{1,n-1}$ be a star. Then $sg_x(K_{1,n}) = n - 2$ or $n - 1$ according as x is an end or non-end vertex, where $n \geq 2$.
(ii) Let $G = K_n$ $(n \geq 2)$ be a complete graph. Then $sg_x(G) = n - 1$ for $x \in G$.

Theorem 2.11. For any vertex x in G, $1 \leq sg_x(G) \leq n - 1$.

Proof. It is clear from the definition of the sg_x-set that $sg_x(G) \geq 1$. Also since the vertex x does not belong to the sg_x-set it follows that $sg_x(G) \leq n - 1$. ■

Remark 2.12. The bounds for $sg_x(G)$ in Theorem 2.11 are sharp. For an even cycle C_{2n}, $sg_x(C_{2n}) = 2$ for any vertex x in C_{2n}. Also for any non-trivial path P_n, $sg_x(P_n) = 1$.
For any end vertex x in P_n. For the complete graph K_n, $sg_x(K_n) = n - 1$ for every vertex x in K_n.

Theorem 2.13. For any integers a, such that $1 \leq a \leq n - 1$, there is a minimal with respect to graph G of order n and a vertex x such that $sg_x(G) = a$.

Proof. If $a = n - 1$ or $n - 2$, $G = K_{1,n-1}$ then the theorem follows from Corollary 2.10 by using $G = K_{1,n-1}$. For $1 \leq a \leq n - 3$, the tree T in Figure 2.2 is provided for$n = k + a$ vertices and it follows from Corollary 2.7 that $sg_x(T) = a$, where x is any non-end vertex of T. As the graph is a tree, it is minimal with respect to edges. ■

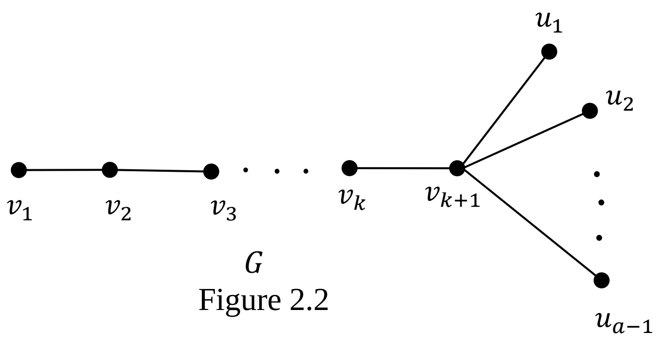


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Figure 2.2

Theorem 2.14. For any graph G, $sg_x(G) = n - 1$ if and only if $degx = n - 1$.

Proof. Let $sg_x(G) = n - 1$. Assume that $degx < n - 1$. Then there is a vertex u in G, such that $ux \notin E(G)$. Since G is connected, there is geodesic from x to u say P with length at least 2. By Note 2.3, x and the internal vertices of n do not belong to the sg_x-set and hence $sg_x(G) \leq n - 2$, which is a contradiction.

Conversely, if $degx = n - 1$, then all other vertices of G are close to x so the sg_x-set is made up of all these vertices. Therefore, $sg_x(G) = n - 1$. ■

Theorem 2.15. Let G be a connected graph. For a vertex x in G, $sg_x(G) = 1$ if and only if x is an end vertex of $P, G = P_n$.

Proof. Let x be an end vertex of P. Then by Corollary 2.8, $G = P_n$. Conversely, let $sg_x(G) = 1$. Then by Corollary

2.8, $sg_x(G) = 1\ \forall x \in V$. Then there exists a vertex y such that every vertex of G is on a diametral path joining x and y. Let $P: x, x_0, x_1, x_2, \dots, x_n = y$ be the fixed x-y geodesic. We prove that $G = P_n$, Suppose not the case. Then there exists $z \in V \setminus V(P_n)$. Then $z \notin \tilde{I}_x[P]$, which is a contradiction. Therefore $G = P_n$. ∎

Theorem 2.16. Let $K_{r,s}$ $(r, s \geq 2)$ be a complete bipartite graph with bipartition (X, Y). Then $sg_x(K_{r,s})$ is s or $s-1$ according as x is in X or x is in Y.

Proof. Case (i) $x \in X$.
Without loss of generality, let $x = x_1$. Since $d(x,y) = 2$ for every $y \in X - \{x\}$, we fix $P_i: x, y_i, x_{i+1}$ $(1 \leq i \leq r-1)$ and so $sg_x(G) \geq r-1$. Let $S = \{x_2, x_3, \dots, x_r\}$. Then the vertices $y_r, y_{r+1}, \dots, y_s$ does not lie on any $x - x_i$ geodesic $(1 \leq i \leq r-1)$. Hence it follows that $S_1 = \{y_r, y_{r+1}, \dots, y_s\}$ is a subset of every vertex strong geodetic set of G and so $sg_x(G) \geq r - 1 + (s - (r-1)) = s$. Let $S_2 = S \cup S_1$. Then S_2 is a vertex strong geodetic set of G so that $sg_x(G) = s$.
Case (i) $x \in Y$.
Without loss of generality, let $x = y_1$. Since $d(x,y) = 2$ for every $y \in Y - \{x\}$, we fix $P_i: x, y_i, x_{i+1}$ $(1 \leq i \leq s-1)$ and so $sg_x(G) \geq s-1$. Let $S = \{y_2, y_3, \dots, y_s\}$. Then S is a vertex strong geodetic set of G so that $sg_x(G) = s-1$. ∎

Theorem 2.17. For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 5)$, $sg_x(W_n) = n-1$ or $n-3$ according as x is K_1 or x is in C_{n-1}.
Proof. Let $V(K_1) = y$ and $V(C_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$. If $x \in V(K_1)$. Then by Theorem 2.14, $sg_x(W_n) = n-1$. Let $x \in V(C_{n-1})$. Without loss of generality, $x = v_1$. Fix $P: v_1, x, v_2$. Since $d(G) = 2$. $S = \{u_3, u_4, \dots, u_{n-2}\}$ is the set of antipodal vertices of x. Then by Observation 2.5 (ii), S is a subset of every vertex strong geodetic set of G and so $sg_x(G) \geq n-4$. Since $y \notin \tilde{I}_x[S]$, S is not a vertex strong geodetic set of G and so $sg_x(G) \geq n-3$. Let $S_1 = S \cup \{y\}$. Then S_1 is a vertex strong geodetic set of G so that $sg_x(G) = n-3$. ∎

Theorem 2.18. For the fan graph $G = K_1 + P_{n-1}$ $(n \geq 5)$

$$sg_x(G) = \begin{cases} n-1 & if\ x \in V(K_1) \\ n-3 & if\ x \in V(P_{n-1}) \end{cases}$$

Proof. Let $V(K_1) = y$ and $V(P_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$.
Case (i) Let $x = y$, Fix $P: v_1, x, v_2$. Let $S = \{v_1, v_2, \dots, v_n\}$. Then S is a set of all eccentric vertices for x. Observation 2.5 (ii) S is a subset of every vertex strong geodetic set of G and so $sg_x(G) \geq n-1$. Since S is a sg_x-set of G we have $sg_x(G) = n-1$. Let $x \in V(P_{n-1})$. Let $x = v_1$. Then $S = \{v_3, v_4, \dots, v_{n-1}\}$ are eccentric vertices of G. By Observation 2.5(ii) S is a subset of every vertex strong geodetic set of G so that $sg_x(G) \geq n-3$. Since S is a sg_x-set of G, we have $sg_x(G) \geq n-3$.
If $x = v_{n-1}$ by the similar way we can prove that $sg_x(G) = n-3$. Let $x \in \{v_2, v_3, \dots, v_{n-2}\}$. Without loss of generality let us assume that $x = v_2$. Then $\{v_1, v_{n-1}\}$ is a set of extreme vertices of G. By Observation 2.5 (i), $\{v_1, v_{n-1}\}$ is a subset of every sg_x-set of G. $\{v_4, v_5, \dots, v_{n-2}\}$ is a set of eccentric vertices of v_2. Then $\{v_4, v_5, \dots, v_{n-2}\}$ is a subset of every vertex strong geodetic set of G and so $sg_x(G) \geq n-3$. Let $S' = \{v_1, v_4, v_5, \dots, v_{n-2}, v_{n-1}\}$. Then S' is a sg_x-set of G so that $sg_x(G) = n-3$.
∎

Theorem 2.19. Let G be a connected graph with k cut vertices. Then every vertex of G is either a cut vertex or an extreme vertex if and only if $sg_x(G) = n-k$ or $n-k-1$ for any vertex x in G.
Proof. Let G be a connected graph in which each vertex falls into one of two categories: a cut vertex or an extreme vertex given that x is not a member of the sg_x-set of G. Observation 2.5 (i) states that $sg_x(G) = n-k$ or $n-k-1$ depending on whether x is a cut vertex or an extreme vertex.
Conversely, suppose that $sg_x(G) = n-k$ or $n-k-1$ for any vertex x in G.
Suppose there is a vertex x in G which is neither a cut vertex nor an extreme vertex. Since x is not an extreme vertex $N(x)$ does not induce a complete subgraph and hence there exist u and v in $N(x)$ such that $d(u,v) = 2$. Also, since x is not a cut vertex of G, $G - \{x\}$ is connected and hence there exists a $u - v$ geodesic say $P: u, u_1, u_2, \dots, u_n, v$ in $G - \{x\}$. Then $P \cup \{v, x, u\}$ is a shortest cycle, say C, that contains both the vertices u and v with length at least 4 in G.
Case 1. Suppose either u or v is not a cut vertex of G. Assume that u is not a cut vertex of G. It is obvious that x is on a $u - v$ geodesic, hence u and x are not part of the sg_x-set. Therefore according to Theorem 2.11, $sg_x(G) \leq n-k-2$, which is a contradiction to the assumption.
Case 2. When u and v are both cut vertices of G. According to Theorem 1.1, there is a division of the set of vertices $V - \{v\}$ into subsets U and W such that the vertex v is on every $u_1 - w_1$ path for vertices $u_1 \in U$ and $w_1 \in W$. Without loss of generality, assume that $x \in U$. Let y be vertex in W with maximum distance from v in W. By choice of y, the vertex y is not a cut vertex of G given that cycle C's order is at least 4, the vertices x and y do not belong to the sg_x-set and hence by Theorem 2.11 $sg_x(G) \leq n-k-2$, which is a contradiction to the assumption. Hence every vertex of G is

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either a cutvertex or an extreme vertex. ■

Corollary 2.20. Let G be a connected block graph with number of cut vertices k. Then for any vertex in G, $sg_x(G) = n-k$ or $n-k-1$.
Proof. Let G be a connected block graph. Then each G vertex is either cut or an extreme vertex and hence by Theorem 2.18, $sg_x(G) = n-k$ or $n-k-1$ for any vertex x in G. ■

Theorem 2.21. If G is a connected of order n and diameter d, then $sg_x(G) \leq n-d+1$ for any vertex x in G.

Proof. For each vertex x in G then $sg_x(G) = n-1 = n-d$ if $G = K_p$. So $G \neq K_p$. Let u and v be two vertices of G such that $d(u,v) = d$ and let $u = v_0, v_1, \dots, v_d = v$ be a u-v geodesic of length d. Now let $S = V(G) - \{v_1, v_2, \dots, v_{d-1}\}$. If $x = v_i$ ($1 \leq i \leq d-1$), then clearly S is an x-vertex strong geodetic set of G so that $sg_x(G) \leq |S| = n-d+1$. If $x = v_i$ ($i = 0, d$), then $S - \{x\}$ is a x-vertex strong geodetic set of G so that $sg_x(G) \leq |S| - 1 = n-d$.

Let $x \neq v_i$ ($0 \leq i \leq d$). Let P and Q be x-v_0 and x-v_d geodesic respectively. Let y be the last vertex common to both P and Q. Let P_1 be the y-v_0 geodesic on P and let Q_1 be the y-v_d geodesic on Q. Let $T = (V(G) - [V(P_1) \cup V(Q_1)] \cup \{v_0, v_d\}$. Then it is clear that T is a x-vertex strong geodetic set of G and so.

$$sg_x(G) \leq n - [d(y,v_0) + d(y,v_d) + 1] + 2$$
$$\leq n - [d(v_0,v_d) + 1] + 2, \text{ by triangle inequality}$$
$$= n-d+1$$

Thus $sg_x(G) \leq n-d+1$ for any vertex x in G. ■

Theorem 2.22. For every non-trivial tree T. Let $sg_x(T) = n-d$ or $n-d+1$ for any vertex x in T if and only if T is caterpillar.
Proof. Let T be any non-trivial tree. Let $P{:}\ u = v_0, v_1, \dots, v_d = v$ be a diametral path. Let k be the number of end vertices of T and l be the number of internal vertices of T other than $v_0, v_1, \dots, v_{d-1}$. Then $d-1+k+k = p$. By Corollary 2.7, $sg_x(T) = k$ or $k-1$ for any vertex x in T and so $sg_x(T) = p-d-l+1$ or $p-d-l$ for any vertex x in T. Hence $sg_x(G) = n-d+1$ or $n-d$ for any vertex x in T if and only if $l = 0$, if and only if all the internal vertices of T lie on the diametral path P, if and only if T is caterpillar. ■

Theorem 2.23. For positive integers r, d and $l \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G with $radG = r$, $diamG = d$ and $sg_x(G) = l$ for some vertex x in G.
Proof. If $r = 1$, then $d = 1$ or 2. If $d = 1$, let $G = K_{l+1}$. Then by Corollary 2.20, $sg_x(G) = l$ for any vertex x in G. If $d = 2$, let $G = K_{1,l}$. Then by Corollary 2.20, $sg_x(G) = l$ for the cut vertex x in G. Now let $r \geq 2$. We construct a graph G with the desired properties as follows.

Case 1. Suppose $r = d$. For $l = 2$. Let $G = C_{2r+1}$. Then $r = d$ and $sg_x(G) = 2$ for any vertex x in G. Now let $l \geq 3$. Let $C_{2r}{:}\ u_1, u_2, \dots, u_{2r}, u_1$ be a cycle of order $2r$. Let G be the graph obtained by adding the new vertices $x_1, x_2, \dots, x_{l-1}$ and joining each x_i ($1 \leq i \leq l$) with u_1 and u_2 of C_{2r}. The graph G is shown in Figure 2.3.

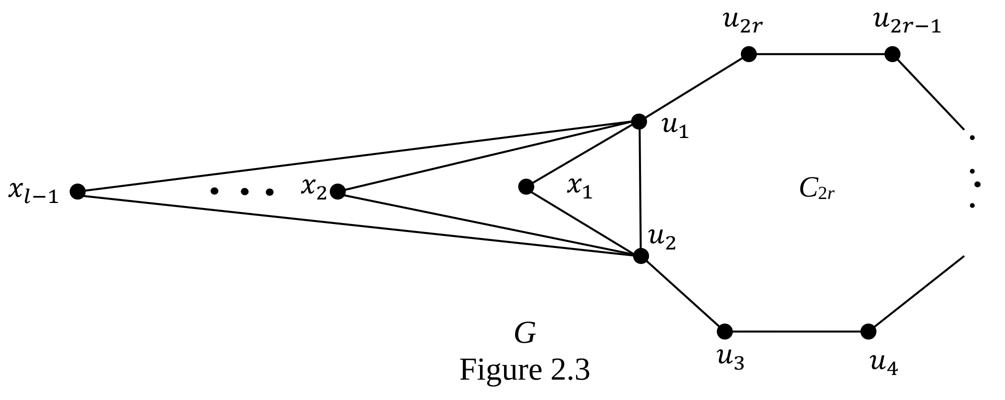

Figure 2.3

It is easily verified that the eccentricity of each vertex of G is r so that $radG = diamG = r$. Fix P: $x_1, u_1, u_2, u_3 \dots, u_{r+1}$. Let $W = \{x_1, x_2, \dots, x_{l-1}\}$ be the set of all extreme vertices of G and let $x = u_{r+1}$. Then by Observation 2.5 (i) W is a subset of every vertex strong geodetic set of G and so $sg_x(G) \geq l-1$. Since W is not a vertex strong geodetic set of G, $sg_x(G) \geq l$. Let $S = W \cup \{u_{r+2}\}$. Then W is a vertex strong geodetic set of G, $sg_x(G) = l$.

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Case 2. Suppose $r < d \leq 2r$. Fix P: $u_{d-r}, u_{d-r-1}, \ldots u_2, u_1, v_1, v_2 \ldots, v_{r+1}$. Let $C_{2r}: v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, u_2, \ldots, u_{d-r}$ be a path of order $d-r+1$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. If $l = 2$, then let $G = H$ let $x = v_{r+1}$. Then $S = \{v_r, u_{d-r}\}$ is a sg_x-set of G so that $l = 2$. If $l \geq 3$, then we add $(l-2)$ new vertices $w_1, w_2, \ldots, w_{l-2}$ to H by joining each vertices w_i $(1 \leq i \leq l-2)$ to the vertex u_{d-r-1} and obtain the graph G of Figure 2.4. Now $radG = r$ and $diamG = d$. Let $W = \{w_1, w_2, \ldots, w_{l-2}, u_{d-r}\}$ be the set of end vertices of G and let $x = v_{r+1}$. Then by Observation 2.5 (i) W is a subset of every vertex strong geodetic set of G and so $sg_x(G) \geq l-1$. Since W is not a vertex strong geodetic set of G, $sg_x(G) \geq l$.
Let $S_1 = W \cup \{v_{r+1}\}$. Then W is a vertex strong geodetic set of G, $sg_x(G) = l$. ∎

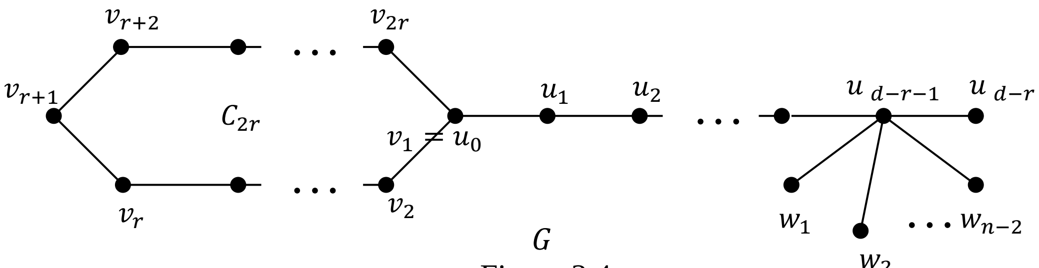

Figure 2.4

Theorem 2.24. For any vertex x in G, $sg(G) \leq sg_x(G) + 1$.
Proof. Let x be any vertex of G and let S_x be a sg_x-set of G lies on an $x-y$ geodesic for some y in S_x. Thus $S_x \cup \{x\}$ is a vertex strong geodetic set of G. Since $sg_x(G)$ is the minimum cardinality of a vertex strong geodetic set, it follows that $sg(G) \leq sg_x(G) + 1$. ∎
Theorem 2.25. For every pair of integers a and b with $1 \leq a \leq b$, there exists a connected graph G such that $g_x(G) = a$ and $sg_x(G) = b$ for some $x \in V(G)$.
Proof. Let $P: x, y, w, z$ be a path on three vertices. Let G be the graph obtained from P by adding the new vertices $z_1, z_2, \ldots, z_{a-1}, v_1, v_2, \ldots, v_{b-a}$ and introducing the edges zz_i $(1 \leq i \leq a)$, zv_i $(1 \leq i \leq b-a)$ and yv_i $(1 \leq i \leq b-a)$. The graph G is shown in Figure 2.5. Let $x = y$.

First we prove that $g_x(G) = a$. Let $Z = \{z_1, z_2, \ldots, z_a\}$ be the end vertices of G. Then by Theorem 1.1 (i), Z is a subset of every g_x-set of G and so $g_x(G) \geq a$. Since Z is a g_x-set of G, $g_x(G) = a$.

Next we prove that $sg_x(G) = b$. We fix the geodesic $P: x, w, z, z_1$ By Observation 2.5 (i), Z is a subset of every sg_x-set of G. It is easily observed that every sg_x- set of G contains each v_i $(1 \leq i \leq b-a)$ and so $sg_x(G) \geq a + b - a = b$. Let $S = Z \cup \{v_1, v_2, \ldots, v_{b-a}\}$. Then S is a sg_x-set of G so that $sg_x(G) = b$. ∎

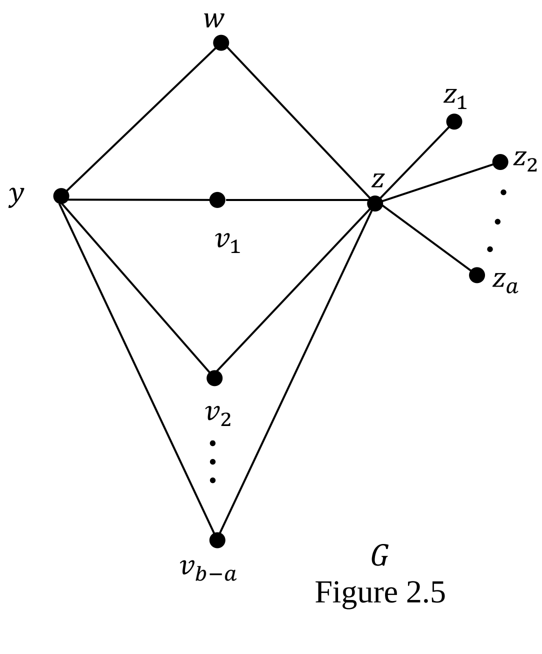

Figure 2.5

3. Conclusions

In this article we explore the concept of the strong geodetic number of a graph. We extend this concept to some other distance related parameters in graphs.

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