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Non-Isomorphic Detour Self-Decomposition Of Graphs

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ABSTRACT

A graph G is is said to have a detour self decomposition $\pi = (G_1, G_2, \ldots, G_n)$ if every subgraph $G_i, 1 \leq i \leq n$ of G have the same detour number as the graph G.Detour self decomposition number of a graph G is the maximum cardinality of the detour self decomposition $\pi = (G_1, G_2, \ldots, G_n)$ and is denoted by $\pi_{sdn}(G)$. If no two detour self-decomposed subgraphs are isomorphic to each other then that decomposition is non-isomorphic detour self-decomposition and maximum cardinality of such decomposition in G is the non-isomorphic detour self-decomposition number of G. Few bounds and some general properties satisfied by this decomposition are studied.

Keywords: Detour number, Detour self-decomposition, Non-Isomorphic Detour self-decomposition, Detour self decomposition number, Non-Isomorphic Detour self-decomposition number

1 Introduction

The graphs $G = (V, E)$ that we have used in this work are all finite, simple, connected and undirected. We refer to [4] for important graph theory terms. G. Chartrand, P. Zhang, and G.L. Johns [1] introduced the notion of the detour number. In the G graph for any two vertices x and y, notation $D(x, y)$ refers to the detour distance which is the longest $x - y$ path of length in G. The $x - y$ detour means a $x - y$ path $D(x, y)$ length. Vertices lying at any $x - y$ detour of G are represented by $I_D[x, y]$ and for any of the subset S of $V(G)$, $I_D[S]$ implies $\cup_{x,y \in S} I_D[x, y]$. If $I_D[S] = V$, S is considered as a detour-set and also the detour set with the least number of vertices in G is the minimum detour set and also the cardinality of this set is a detour number.

Definition 1.1 *[6] The edge disjoint subgraphs collection $G_1, G_2, \ldots, G_n$ of G represents the decomposition of G if G's each edge is in exactly one $G_i, 1 \leq i \leq n$.*

Theorem 1.2 *[2] Every detour-set for the non-trivial connected graph G consists every end vertices of that graph.*

Theorem 1.3 *[2] A tree T having k end-vertices has detour number k.*
The idea behind H-decomposition was introduced by L.Posa, P. Erdos, and A. W. Goodman [3] and various problems related to H-decomposition has been studied in recent years. E.E.R. Merly & Anlin Bena E introduced the concept "Detour self-decomposition of graphs"[5].

Definition 1.4 *[5] The decomposition $\Pi = (G_1, G_2, \ldots, G_n)$ of G is stated to be detour self-decomposition if $dn(G) = dn(G_i)$,$1 \leq i \leq n$ and detour self-decomposition number of G is the greatest cardinality of such decomposition and is represented as $\pi_{sdn}(G)$.*

In this paper we introduce the concept of Non-isomorphic Detour self-decomposition of graphs.

2 Main Results

Definition 2.1 *The decomposition $\Pi = (G_1, G_2, \ldots, G_n)$ of G is said to be a non-isomorphic detour self-decomposition if $dn(G) = dn(G_i)$,$1 \leq i \leq n$ and any pair of distinct subgraphs from Π is non-isomorphic to each other. The maximum cardinality of such Π is said to be the non-isomorphic detour self-decomposition number of G and is denoted as*

$\pi_{nsdn}(G)$.

Example 2.2

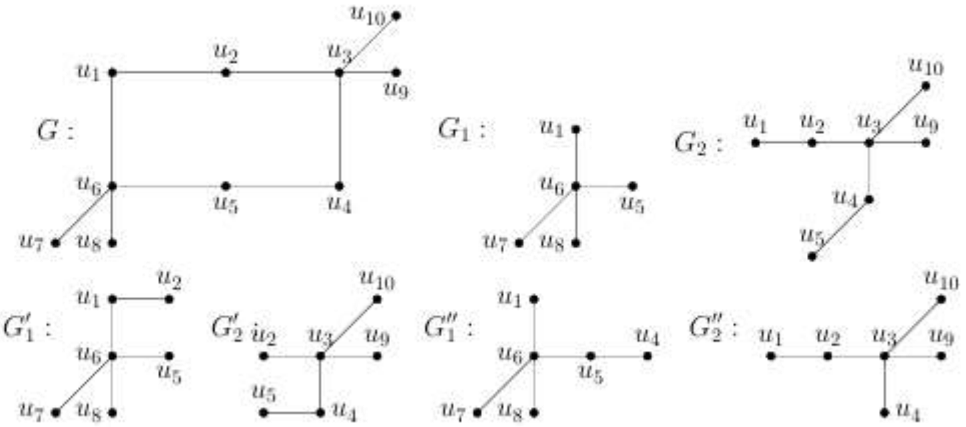

Figure 2.1: A graph G and its decompositions Π_1, Π_2, Π_3

Consider the graph G, the detour set of G is $\{u_7, u_8, u_9, u_{10}\}$ and this set is minimum, hence $dn(G) = 4$. This graph G can be detour self-decomposed into the following three ways:$\Pi_1 = (G_1, G_2), \Pi_2 = (G_1', G_2',), \Pi_3 = (G_1'', G_2'')$.
Since the graphs $G_1, G_2, G_1', G_2', G_1'', G_2''$ are all trees with exactly four pendant vertices, from theorem 1.3, its detour number is 4. The subgraphs G_1' and G_2' are edge-disjoint but are isomorphic to each other and the subgraphs G_1'' and G_2'' are edge-disjoint but are isomorphic to each other. Thus the decompositions Π_2 and Π_3 are detour self-decompositions of G but are not non-isomorphic. In this graph Π_1 is the only non-isomorphic detour self-decomposition.

Theorem 2.3 *The path graph P_p has non-isomorphic detour self-decomposition number n for all $n \in \mathbb{N}$ if and only if* $\frac{n(n+1)}{2} + 1 \leq p \leq \frac{(n+1)(n+2)}{2}$.

Proof. Suppose a path P_p has non-isomorphic detour self-decomposition number n *.i.e.,* $\pi_{nsdn}(P_p) = n$.
In order to find the maximum possible such non-isomorphic detour self-decomposition $\Pi = (G_1, G_2, \ldots, G_n)$ the subgraphs must be decomposed in such a way that it must have least number of edges at the same time no two subgraphs can have same number of edges.
One such possibility is $|E(G_i)| = i, 1 \leq i < n$ and $n \leq |E(G_n)| \leq 2n$.
Since $|E(P_p)| = |E(G_1)| + |E(G_2)| + \ldots + |E(G_n)|$, we get $\frac{n(n+1)}{2} \leq |E(P_p)| \leq \frac{n^2+3n}{2}$.
Therefore $\frac{n(n+1)}{2} + 1 \leq p \leq \frac{(n+1)(n+2)}{2}$.
Conversely, assume that $\frac{n(n+1)}{2} + 1 \leq p \leq \frac{(n+1)(n+2)}{2}$.
First consider the path $P_{\frac{n(n+1)}{2}+1}$.
Then the path $P_{\frac{n(n+1)}{2}+1}$ can be decomposed by taking G_i as the graph induced by the vertices $\{v_{\frac{(i-1)i}{2}+1}, v_{\frac{(i-1)i}{2}+2}, \ldots, v_{\frac{i(i+1)}{2}+1}\}$, $1 \leq i \leq n$.
Clearly $\Pi = (G_1, G_2, \ldots, G_n)$ is a non-isomorphic detour self-detour decomposition of $P_{\frac{n(n+1)}{2}+1}$.
Hence $\pi_{nsdn}(P_{\frac{n(n+1)}{2}+1}) = n$.
Now, let us consider paths P_p where $\frac{n(n+1)}{2} + 1 < p \leq \frac{(n+1)(n+2)}{2}$.
Any such path has $P_{\frac{n(n+1)}{2}+1}$ as a subgraph.
So considering the first $\frac{n(n+1)}{2} + 1$ vertices of P_p as the subgraph $P_{\frac{n(n+1)}{2}+1}$ and decomposing it same as above and removing $G_1, G_2, \ldots, G_n$ from P_p results into a path $P_j, 1 < j \leq n$.
But the path P_j and its subgraphs are always isomorphic to any one of $G_1, G_2, \ldots, G_n$.
Hence taking G_n' as the subgraph induced by the vertices
$\{v_{\frac{(n-1)n}{2}+1}, v_{\frac{(n-1)n}{2}+2}, \ldots, v_{\frac{n(n+1)}{2}+1}, \ldots, v_p\}$ in P_p and remaining subgraphs $G_1, G_2, \ldots, G_{n-1}$ same as the decomposition of

$P_{\frac{n(n+1)}{2}+1}$, we get the desired decomposition.
In this way each subgraph is non-isomorphic to each other and except G_n' all other G_i,s have minimum possible edges hence this decomposition is maximum.
Hence the theorem. ∎

Theorem 2.4 *For any* $m, n \in \mathbb{Z}^+$ *and* $m, n > 2$, *if a graph* G *has* $dn(G) = m$ *and* $\pi_{nsdn}(G) = n$ *then* $|E(G)| \geq n(m+2) - 3$.

Proof. Let $m, n \in \mathbb{Z}^+$ and $m, n > 2$.
Consider a graph G with $dn(G) = m$ and $\pi_{nsdn}(G) = n$.
Let $\Pi = (G_1, G_2, \ldots, G_n)$ be a non-isomorphic detour self-decomposition of G.
Since the subgraphs are non-isomorphic to each other, without loss of generality let us assume that $|E(G_1)| \leq |E(G_2)| \leq \ldots \leq |E(G_n)|$.
We know that S_m is the only graph with least number of edges and detour number m for all $m > 2$. Hence each $|E(G_i)| \geq m$ for all $1 \leq i \leq n$.
Also, there exists exactly one spider tree upto isomorphism with m legs that has exactly $m+1$ edges and detour number m which is non-isomorphic to S_m.
For graphs with edges greater than $m+1$, there exists more than one graph upto isomorphism with detour number m and non-isomorphic to S_m.
Hence $|E(G_1)| \geq m, |E(G_2)| \geq m+1$ and $|E(G_j)| \geq m+2, 3 \leq j \leq n$.
Since $|E(G)| = |E(G_1)| + |E(G_2)| + \ldots + |E(G_n)|$, we get
$|E(G)| \geq n(m+2) - 3$. ∎

Theorem 2.5 *Let* G *be a graph with detour number* 2. *If* G *has continuous monotonic path decomposition* $\Pi = (G_1, G_2, \ldots, G_n)$, *then* Π *is a non-isomorphic detour self-decomposition of* G.

Proof. Let G be a graph with $dn(G) = 2$.
Assume that G has continuous monotonic path decomposition $\Pi = (G_1, G_2, \ldots, G_n)$.
By definition, $|E(G_i)| = i, 1 \leq i \leq n$.
Therefore, any two subgraph from Π is always non-isomorphic to each other.
Since each $G_i, 1 \leq i \leq n$ is a path, $dn(G_i) = 2, 1 \leq i \leq n$.
Hence Π is a non-isomorphic detour self-decomposition of G. ∎

Theorem 2.6 *For any graph* G, $1 \leq \pi_{nsdn}(G) \leq \pi_{sdn}(G)$.

Proof.
If a graph G cannot be further be decomposed into two or more subgraphs satisfying the conditions for non-isomorphic detour self-decomposition, then $\pi_{nsdn}(G) = 1$. Otherwise $\pi_{nsdn}(G) > 1$.
Hence $\pi_{nsdn}(G) \geq 1$.
Suppose the graph G has $\pi_{sdn}(G) = n$.
Then there exists a $\Pi = (G_1, G_2, \ldots, G_n)$ such that $dn(G) = dn(G_i)$, for all $1 \leq i \leq n$.
If no two G_i and G_j where $1 \leq i, j \leq n$ and $i \neq j$ are isomorphic to each other then this Π is a non-isomorphic detour self-decomposition of G. In this case $\pi_{nsdn}(G) = \pi_{sdn}(G)$.
Otherwise G may or may not be decomposed into more than two subgraphs which are not isomorphic to each other and detour number is same as G. Then $\pi_{nsdn}(G) < \pi_{sdn}(G)$.
Hence $\pi_{nsdn}(G) \leq \pi_{sdn}(G)$.

Therefore, $1 \leq \pi_{nsdn}(G) \leq \pi_{sdn}(G)$. ∎

Theorem 2.7 *For a tree* $G, \pi_{nsdn}(G) \geq 2$ *if and only if* $|V(G)| \geq 4$ *and* $dn(G) = 2$.

Proof. Let G be a tree.
Assume that G admits non-isomorphic detour self decomposition and $\pi_{nsdn}(G) \geq 2$. Let this decomposition be $\Pi = (G_1, G_2, \ldots, G_n)$ where $n \geq 2$.
Since subgraph of each $G_i, 1 \leq i \leq n$ is a tree and for a tree its minimum detour set is its pendant vertices, G_i has $dn(G_i)$

number of pendant vertices.
Each pendant vertex of $G_i, 1 \leq i \leq n$ is either a pendant vertex or an inner vertex in G.
From these subgraphs, we choose a subgraph that contains maximum number of pendant vertices that are inner vertices of G. Without loss of generality, let this graph be G_1 with n_1 pendant vertices which are inner vertices in G. Clearly $1 \leq n_1 \leq dn(G_1)$
Since detour number of G is the cardinality of set of pendant vertices, we calculate the pendant vertices of G by using G_1.
Since G is a tree, the subgraphs attached from these n_1 vertices are disjoint from each other. In G, if a pendant vertex of a subgraph G_j(say) is also a pendant vertex of G_1, then this subgraph contribute atleast $dn(G_j) - 1$ pendant vertices.
In this way n_1 inner vertices of G_1 contribute $dn(G_j) - 1$ pendant vertices of G where $j \in \{1,2,3,\dots,n\}$.
The remaining $dn(G_1) - n_1$ pendant vertices of G_1 contribute exactly one pendant vertex to G.
By definition, $dn(G) = dn(G_i), 1 \leq i \leq n$.

$$\begin{aligned}\text{Numberofpendantverticesof} G &\geq (dn(G) - 1)n_1 + (dn(G) - n_1)\\ &= dn(G)(n_1 + 1) - 2n_1\end{aligned}$$

But, number of pendant vertices of $G = dn(G)$
Then, $dn(G) \geq dn(G)(n_1 + 1) - 2n_1$
Therefore $dn(G) \leq 2$.
But, by theorem , $dn(G) \geq 2$.
Thus $dn(G) = 2$.
Since $dn(G) = 2$, G is path.
Given every subgraph in Π is non-isomorphic to each other, therfore without loss of generality, let us assume that $|E(G_1)| < |E(G_2)| <\dots< |E(G_n)|$.
Assuming the maximum possible decomposition in G, we get
$|E(G_i)| = i, 1 \leq i \leq n - 1$ and $n \leq |E(G_n)| \leq 2n$.
Since $|E(G) = |E(G_1)| + |E(G_2)| + \dots + |E(G_n)|$, we get
$\frac{n(n+1)}{2} \leq |E(G)| \leq \frac{n(n+1)}{2} + n$.
Therefore,

$$\frac{n(n+1)}{2} + 1 \leq |V(G)| \leq \frac{n(n+1)}{2} + n + 1. \qquad (1)$$

From equation 1 taking the lower bound and substituting the value $n \geq 2$, we get $|V(G)| \geq 4$.
Conversely,assume that $|V(G)| \geq 4$ and $dn(G) = 2$.
Since G is a tree and $dn(G) = 2$, G is a path.
From theorem 2.3, if $|V(G)| = 4, \pi_{nsdn}(G) = 2$.
Therefore for $|V(G)| \geq 4, \pi_{nsdn}(G) \geq 2$.
Hence the theorem. ■

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