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Isomorphism Challenges In Groups With Restricted Centers And Exploring Narrow Normal Subgroups In Coxeter Groups And Their Automorphisms

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Abstract:

The modular isomorphism problem for groups with a centre of index at most P^3, and the study of narrow normal subgroups within Coxeter groups and their automorphism groups, presents a rich field of investigation ripe with intriguing challenges and potential insights. The major contribution of this paper is addressing the modular isomorphism problem in groups with a centre of index at most P^3 while concurrently exploring narrow normal subgroups within Coxeter groups and their automorphisms. This interdisciplinary inquiry delves into the structural complexities and symmetrical intricacies inherent in both areas of group theory, aiming to unveil connections and insights across modular isomorphism and the dynamics of narrow subgroups within Coxeter groups.

Keywords. Modular isomorphism problem, Small group algebra, Coxeter groups, Narrow normal subgroups, Abelian P - groups.

1.Introduction

The intersection of group theory, particularly concerning the modular isomorphism problem for groups with a centre of index at most P^3, and the study of narrow normal subgroups within Coxeter groups and their automorphism groups, presents a rich field of investigation ripe with intriguing challenges and potential insights. Understanding the isomorphism properties of groups with constrained centres is fundamental to comprehending their structural intricacies and their role in various mathematical contexts. Concurrently, delving into the narrow normal subgroups of Coxeter groups and their automorphisms provides a deeper understanding of their internal dynamics and symmetries. This interdisciplinary exploration aims to uncover connections between these seemingly distinct areas of group theory, offering new perspectives and avenues for further exploration in both modular isomorphism and the study of narrow subgroups within Coxeter groups.

2 . The Modular isomorphism problem for groups with a centre of index at most P^3

Definitions:
***2.1: Coxeter Groups*:**
The Mathematical structures with specific rules and weself interested in how they transform themselves, known as their automorphism groups.
Coxeter Groups have unique properties and rules for combaining elements also understanding these rules is crucial for studing how Coxeter groups change.
2.2 : The Small Group Algebra:
Let f denote the field with P elements for a prime number $P > 2$ and let g be a finite P-group. In this section, derive results on the small group algebra that will later be used to solve the modular isomorphism problem for groups with a center of index P^3.
In particular, let $fg \to s$ be the natural projection onto the small group algebra s and let $\mathrm{V}(s) = 1 + \mathrm{I(g)}/[\mathrm{I(g)I}(\lambda 2(\mathrm{g}))]$ denote the group of units of augmentation 1 of s. Furthermore, let A be a subgroup of $V(s)$, so $V(s) = \bar{g}A$.
2.3 : Elementary abelian P-groups:
An elementary abelian group is an abelian group in which all elements other than the identity have the same order. This common order must be a prime number, and the elementary abelian groups in which the common order is p are a particular kind of p-group is knownas abelian P - groups.

2.4 : Narrow Normal Subgroup:
A Group is called narrow if it does not contain a copy of an non – abelian free group. We describe the structure of finite and narrow normal subgroup in Coxeter Groups and their automorphism groups.

Theorem : 2.1
Let P be an odd prime number and let f be the field with P elements. Consider finite P-groups f and H with $fg \cong fH$ and let $|g/\emptyset(g)Z(G)| = P^d$. If

a) $g^P \cap \lambda 2(g) \subseteq \lambda 2(g)^P \lambda 3(g)$

b) $|\lambda 2(g)/\lambda 2(g)^\wedge P\ \lambda 3(g)| = P^{\frac{d}{2}}$,

then $(g)/\lambda 2(g)^P \lambda 4(g) \cong H/\lambda 2(H)^P \lambda 4(H)$.

Proof :
Consider the isomorphism of elementary abelian P-groups

$$\Psi: g/\emptyset(g) \to I(g)/I(g)^2, X\emptyset(g) \to (X-1) + I(g)^2$$

It has an f-vector space decomposition $Z(fg) = fZ(g) + Z(fg) \cap K(fg)$
It is well-known that $fg.K(fg) = K(fg).fg = I(\lambda 2(g))fg$
As $\lambda 2(g) \subseteq \emptyset(g) = (1 + I(g)^2) \cap g$ holds,
it have $K(fg) \subseteq I(\gamma 2(g))fg \subseteq I(g)^2$
In particular, $\propto$ restricted to an isomorphism

$$Z(g)\emptyset(g)/\emptyset(g) \to (Z(fg) \cap I(G) + I(g)^2)/I(g)^2$$

Of course, the analogous reasoning applies to H. In particular, thus obtain

$$P^d = |g/\emptyset(g)Z(g)| = |I(g)/(Z(fg) \cap I(g) + I(g)^2| = |H/\emptyset(H)Z(H)|$$

Moreover, note that the conditions (a) and (b) depend only on the quotient $g/\lambda 2(g)^P \lambda 2(g)$,
which is determined by f g. Thus, if g satisfies (a) and (b), then the analogous statements hold for H.
Consider the group V and the projection $\pi: V(fg) \to V$ introduced. Let $\{g1, g2, \dots, gk, c1, \dots, cm\}$ be a set of generators of $\pi(g)$ satisfying relations.

$$R(n, \alpha, \beta)(g1, \dots, gk, c1, \dots, cm)$$

For $1 \le i \le k$, the decomposition $V = \pi(H)e$ guarantees that $j_i = h_i e_i$ for some unique $h_i \in \pi(H)$ and $e_i \in E$. Moreover $c_j \in \lambda 2(\pi(g)) = \lambda 2(s) = \lambda 2(\pi(H))$ for $l = 1, \dots\dots, m$.
Now let $d < i \le K$, so $j_i \in Z(\pi(g))$.
It has

$$(j_i - 1) + I(g)^2 = (h_i e_i - 1) + I(g)^2 = h_i - 1 + e_i - 1 + I(g)^2 = (h_i - 1) + I(g)^2$$

using $e_i - 1 \in I(g)^2$. In particular, this yields $(h_i - 1) + I(g)^2 \in (Z(fg) \cap I(g) + I(g)^2/I(g)^2$. Using the isomorphism, and the analogue for H, derive that $h_i = \hat{h}_i \tilde{h}_i$, with $\hat{h}_i \in Z(\pi(H))$ and $\tilde{h}_i \in \emptyset(\pi(H)) = \lambda 2(\pi(H))\pi(H)^P$. Observe that $\pi(H)^P$ is central in $(\pi(H))$, so furthermore assume that $\tilde{h}_i \in \lambda 2(\pi(H))$. Then $\{h_1, \dots, h_d, \hat{h}_{d+1}, \dots\dots, \hat{h}_K, c_1, \dots, c_m\}$ forms a generating set of $\pi(H)$. Now show that it satisfies the relations $R_{(n,\alpha,\beta)}(h_1, \dots, h_d, \hat{h}_{d+1}, \dots\dots, \hat{h}_K, c_1, \dots, c_m$.
Then H is an epimorphic image of g of the same size and we obtain $g \cong H$ as desired. For $1 \le i \le k$, one has $j_i^P = (h_i e_i)^P = h_i^P e_i^P [h_i e_i]^P = h_i^P$,
and, in particular, for $d < i \le K$, one has $j_i^P = h_i^P = \hat{h}_i^P \tilde{h}_i^P = \hat{h}_i^P$. Then condition (1) follows. Condition (2) also follows readily because for $1 \le i \le k$ and arbitrary $X_i, X_l, X_k, \in A$, we have that $[[j_i X_i, j_l X_l], j_k X_k] = [[j_i, j_l][X_l, j_l][j_i, X_l], j_k X_k = [[j_i j_l], [j_k X_k] = [[j_i, j_l], j_k$.
This finalizes the proof.

3. Groups with a center of index P^3:
Let P be a prime number. In this section, use the results of the small group algebra to give a positive answer to the modular isomorphism problem for groups with the center of odd index P^3.
Lemma 3.1 : Let P be an odd prime number and let g be a finite P-group with $|g: Z(g)| = P^3$. Then:
(1) The nilpotency class of g is at most 3.
(2) $\lambda 2(g)$ is elementary abelian.
(3) Either g has nilpotency class 2 or $|g/\emptyset(g)Z(g)| = P^2$

Proof:

Examination of the upper central series yields that the nilpotency class of g is at most 3. The quotient $g/Z(g)$ is isomorphic to one of the five groups of order P^3, namely to one of $M_{P^3}, M_P^3, M_{P^2} \times M_P, (M_P \times M_P)M_P$, or $M_{P^2}M_P$. Observe that (g)Z(g) is not cyclic as g is abelian otherwise.

Next, we show that $\lambda 2(g)$ is elementary abelian. It is clear that $\lambda 2(g)$ is abelian as $[\lambda 2(g), \lambda 2(g) \subseteq \lambda 4(g) = 1.$ Thus it suffices to show that the generators of $\lambda 2(g)$ have order P. Note that $\lambda 2(g)$is elementary abelian. Indeed, it is generated by elements of the form [X, Y] with $X \in g$ and $Y \in \lambda 2(g)$; since $\lambda 2(g)Z(g)/Z(g)$ has order at most P, and have $YP \in Z(g)$, so shows that $1 = [X, YP] = [X, Y]P.$ Let $a, b_1, b_2 \in g$ be such that $\frac{g}{Z(g)} =< aZ(g), b_1 Z(g), b_2 Z(g) >$. By the structure of $g/Z(g)$, assume that $b_i^P \in Z(g)$ for i=1,2. Then for every $X \in g$ and $i \in \{1,2\}$, Lemma 3.1 yields that

$$1 = [X, b_i^P] = [X, b_i]^P [[X, b_i], b_i]^{(P\ 2)} = [X, b_i]^P$$

This shows that $\lambda 2(g)$ is elementary abelian. Finally, if g has nilpotency class 3, then $g/Z(g)$is isomorphic to one of the two non-abelian p-groups of order P^3. Then it is clear that $\lambda 2(g)Z(g)/Z(g) = P$, and hence $|g/\emptyset(g)Z(g)| = P^2$

Theorem 3.2:
Let P be an odd prime number, let f be the field with P elements, and let g and H be finite P-groups. Suppose that $|g: Z(g)| = P^3$. If $fg \cong fH$, then $g \cong H$.
Proof:
The $\lambda 2(g)$ is elementary abelian. If the nilpotency class of g is 2, then $g \cong H/\lambda 2(H)^P \lambda 3(H)$. so $g \cong H$ due to $|g| = |H|$. Thus, assume that the nilpotency class of g is 3, and that $|g/\emptyset(g)Z(g)| = P^2$.

4. Remarks

- Point out that there cannot be an analogue of Theorem 3.2 for P = 2, non-isomorphic finite 2-groups with centers of index 8 and isomorphic group algebras over every field of characteristic 2 are presented. Hence the result underlines the difference between the cases P = 2 and P > 2 for this problem.
- Observe that yields that the finite p-groups (P odd) with extraspecial central quotient are exactly the groups with center of index P^3 and nilpotency class 3. Thus, Theorem 6 gives a positive answer to the modular isomorphism problem for this class of groups [1].

5. Narrow normal subgroups of the automorphism group of a Coxeter group

The main aim is to determine the finite normal subgroups and the narrow normal subgroups of Coxeter Group.

Corollary 5.1
Let C be a Coxeter group.
(1) If $C_{sph} = \{1\}$, then C does not contain any non-trivial finite normal subgroup.
(2) If $C_{sph} = C_{aff} = \{1\}$, then C does not contain any non-trivial narrow normal subgroup.
At the end of this paper, to point out that the property of a group not having non-trivial finite normal subgroups is inherited by finite index normal subgroups.
Corollary 5.2
Let C be a Coxeter group and let H_s be a normal subgroup of Aut(C).
(1) Assume that C has no spherical component, that is, $C_{sph} = \{1\}$. Then Aut(C) has no nontrivial finite normal subgroup, and H_s is narrow if and only if H_s is a subgroup of $Aut(C_{aff})$.
(2) Assume that C has no spherical component and no affine component, that is, $C_{sph} = C_{aff} = \{1\}$. Then Aut(C) has no non-trivial narrow normal subgroup.

6. Automatic continuity for Coxeter groups and their automorphism groups

The application of Corollary 5.1 (i) and Corollary 5.2 (ii) in the direction of automatic continuity of Coxeter groups and their automorphism groups.
Given a map $\oint : L_H \to G_d$ between a locally compact Hausdorff group L_H and a discrete group G_d, the automatic continuity problem is the following: assuming $\oint$ is a group homomorphism on the level of groups, find conditions on G_d or $\oint$ which imply that $\oint$ is continuous.
A discrete group G_d is called $l_c H_s$ −slender if every group homomorphism $\oint : L_H \to G_d$ on the level of groups (i.e. abstract group homomorphism) where L_H is a locally compact Hausdorff group is continuous. Many groups are known to be $l_c H_s$ −slender, for example, free and free abelian groups.

The discrete group G_d almost l_cH_s −slender if every abstract surjective group homomorphism $\oint : L_H \rightarrow G_d$ from a locally compact Hausdorff group L_H onto G_d is continuous. An algebraic description of almost l_cH_s −slender groups was proven. Here recall a weaker version of this theorem which is suitable for Coxeter groups and their automorphism groups.

Theorem 6.1:
Let $\oint : L_H: G_d$ be an epimorphism from a locally compact Hausdorff group L_H to a countable group G_d. If every torsion subgroup of G_d is finite, G_d does not contain q and G_d does not have non-trivial finite normal subgroups, then $\oint$ is continuous.
Combining Corollary 5.1 (i) with Theorem 6.1 obtain a characterization of almost l_cH_s −slender Coxeter groups.
Corollary 6.2
Let C be a Coxeter group. Decompose C as a direct product $C_{gen} \times C_{aff} \times C_{sph}$. Group C is almost l_cH_s −slender if and only if $C_{sph} = \{1\}$.
Proof:
If C_{sph} is non-trivial, then there exists always a discontinuous surjective group homomorphism from the compact group $L_H := qNC_{sph}$ into C_{sph}. Denote this epimorphism by $\oint_{sph}$. Thus the group homomorphism.

$$\oint : C_{gen} \times C_{aff} \times L_H \rightarrow C_{gen} \times C_{aff} \times C_{sph}$$

where $\oint$ is identity on $C_{gen} \times C_{aff}$, and $\oint_{L_H} := \oint_{sph}$ is a surjective discontinuous group homomorphism.
For the other direction, since C is a finitely generated linear group know that torsion subgroups in C are finite. Further, since C is residually finite, C cannot have q as a subgroup. Thus, if $C_{sph} = \{1\}$. Corollary 5.1 (i) implies that C does not have non-trivial finite normal subgroups. Hence by Theorem 6.1, know that C is almost l_cH_s −slender.
To show similar results for automorphism groups Coxeter groups need the knowledge that torsion subgroups in these automorphism groups are always finite.
Recall, a group G_d is called residually p-finite for a prime number P if for every $g \in G_d, g \neq 1$ there exists an epimorphism $\oint_p: G_d \rightarrow F_p$ where $F_p: G_d \rightarrow F_p$ where F_p is a finite P-group and $\oint_p(g) \neq 1$. A direct consequence of residually P-finiteness is that every finite order element in a residually P-finite group has P-power order. Thus, if G_d is residually p-finite and residually q-finite for $P \neq q$, then G_d is torsion-free. Further, a group G_d is called virtually residually P-finite if there exists a subgroup $H_s \subseteq G_d$ of finite index which is residually P-finite.

7. Conclusion

The modular isomorphism problem for groups with a center of index at most P^3 was solved by primarily leveraging results derived from small group algebra. Through intricate theorems and proofs, conditions were established under which finite *P*-groups were isomorphic based on specific properties of their quotients and central quotients. This section provided a robust framework for addressing the modular isomorphism problem, especially for groups with a center of odd index P^3. Meanwhile, the study of automorphism groups of Coxeter groups elucidated the properties of narrow normal subgroups and finite normal subgroups. By establishing corollaries and theorems, light was shed on the automatic continuity of Coxeter groups and their automorphism groups, unveiling conditions under which these groups exhibited slender characteristics. Furthermore, it was established the virtually torsion-free nature of the automorphism groups of Coxeter groups, enabling insights into their slender properties. Collectively, these sections contributed significantly to understanding the structural intricacies and automorphism properties of finite groups and Coxeter groups, offering valuable insights into their algebraic and geometric properties.

8. References

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