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Halfway Number Of Graphs

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Abstract:

In this paper we have introduced the concept of halfway distance in graphs and the halfway number. The idea of finding the halfway distance has been developed while working with the distance parameter detour distance in graphs. The concept of halfway distance can be used to calculate the median distance in various distance locating problems. The distance found using the above concept is effective in all aspects. Let K be the set of all detour distance between every pair of vertices of G. The halfway distance $H(u, v)$ is the median of K. A path of length atmost $H(u, v)$ is called the halfway path. A set $S \subseteq V(G)$ is called a halfway set of G ifevery vertex of G is contained in a halfway path joining some pair of vertices of S. The halfway number is the minimum order of the halfway set and is denoted by $hn(G)$.

Keywords: detour distance, median, halfway distance, halfway path
AMS Classification: 05C12,05C69

1.Introduction

In graph theory, distance between any two vertices of a graph is nothing but the minimum number of edges joining the two vertices. Using the distances between two vertices in a graph, we can calculate the eccentricity, diameter, radius, center of the graph. The interest in finding various distances is unstoppable. It next moves on to find the shortest distance and longest distance. The shortest distances in graphs has major application in mapping, routing, computer networks, etc,…Various studies were still in progress related to the shortest distances. Algorithms like Dijkstra's algorithm, Floyd-Warshall algorithm were used to find the shortest path. The term geodesic has been used to denote the shortest path. The concept of geodesic is also an important topic in graph. In the same way, the longest distance between any two points is called the detour distance. If we consider the distance parameter, both geodesic and detour distance has similar significance.It has various application in networking, routing etc,....For instance, let us consider the task of fixing the bus stops traversing villages/cities. It is important that every person should be able to use the travelling facility so that no person walks too long for bus stoppings. Here we can use the notion of halfway distance. One can find the longest distance between each village/city. Then find the halfway distance between villages/cities. Using this distance we can fix the stopping which will be beneficial.

Definition 1.1. Let K be the set of all detour distance between every pair of vertices of G. The halfway distance $H(u, v)$ is the median of those possible detour distances. A path of length atmost $H(u, v)$ is called the halfway path. A set $S \subseteq V(G)$ is called a halfway set of G if every vertex of G is contained in a halfway path joining some pair of vertices of S. The halfway number is the minimum order of the halfway set and is denoted by $hn(G)$.
Example:

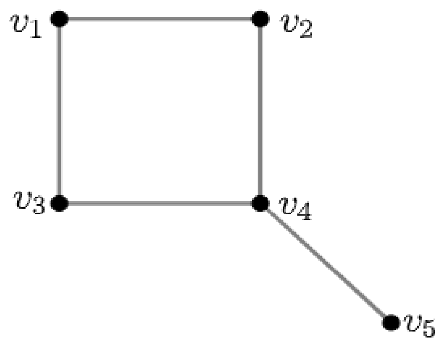

Fig.1

Let K={3,3,2,3,2,3,4,3,2,1} be the set of all detour distances of the vertices of the graph. Halfway distance of the graph is med({3,3,2,3,2,3,4,3,2,1})=3. Now the paths $v_1 - v_2, v_1 - v_3, v_1 - v_5, v_2 - v_4, v_3 - v_4$ are halfway paths because the detour distance between these paths satisfies the halfway distance. Therefore, S={v_1, v_5}is a minimum halfway set of the graph in fig.1 and hence the halfway number $hn(G) = 2$.

1.2.Steps for finding the halfway distance

Let G be a connected graph of order n. Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of G. To find the halfway distance of a graph G we need to follow the following steps.

Step 1: Start with v_1. Find the best possible detour distances between v_1 and v_2 and denote it by d_{12}. Next find the detour distances between v_1 and v_3. Name such a distance by d_{13}. Similarly find the detour distances between v_1 and every other vertices of G. Let those be denoted as $d_{12}, d_{13}, \dots, d_{1n}$.
Step 2: Next start with v_2. Find the detour distance between v_2 and v_3. Denote such distance by d_{23}. Similarly find all the detour distance between v_2 and v_4, v_2 and $v_5, \dots, v_2$ and v_n. Let it be denoted as $d_{23}, d_{24}, \dots, d_{2n}$.
Step 3: Continuing this way find all the detour distance between v_3 and $v_4, \dots, v_n$, v_4 and $v_5, \dots, v_n$ and between v_{n-1} and v_n. Let all those distances be denoted as $d_{34}, d_{35}, \dots, d_{3n}, d_{45}, d_{46}, \dots, d_{4n}, \dots, d_{(n-1)n}$.
Step 4: Let H be the collection of all such detour distances. Then H = $\{d_{12}, d_{13}, \dots, d_{1n}, d_{23}, d_{24}, \dots, d_{2n}, d_{34}, d_{35}, \dots, d_{3n}, d_{45}, d_{46}, \dots, d_{4n}, \dots, d_{(n-1)n}\}$. This collection H will contain nC_2 elements.
Step 5: Arrange the elements of H in an ascending order so that always $d_{ij} \leq d_{kl}$ where $1 \leq i, j, k, l \leq n$. Now find the median of H. If the number of elements in the set H is odd, then the middle term will be the median. If the number of elements in the set H is even, then the average of $\frac{nC_2}{2}$ th term and $\frac{nC_2}{2} + 1$ th term will be the median.
Step 6: The median of H found in step 5 is the halfway distance of G and it is denoted by $H(u, v)$.

Thus by using the above steps one can find the halfway distance of G.

1.3.General Bound Halfway distance ranges from 1 to n-1. Halfway distance attains its upper bound for complete graph and it attains its lower bound for P_2 and P_3.

Theorem 1.4. $H(u, v) = n - 1$ for the complete graph K_n.

Proof. Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of K_n. Since each vertex $v_i, 1 \leq i \leq n$ is adjacent to every other vertices of G, there exists a cycle of length of n. The detour distance between any two vertices of a complete graph of order n is n-1. Halfway distance $H(u, v)$ is the median of those detour distances. Thus we got $H(u, v) = n - 1$.

Theorem 1.5. For path graph, $H(u, v) < n - 1$.
Proof. Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of the path graph where v_1 and v_n are the pendant vertices. The detour distance between v_1 and v_n is $n - 1$. The detour distance between v_2 and v_n is $n - 2$. The detour distance between v_3 and v_n is $n - 3$.…………………The detour distance between v_{n-1} and v_n is 1. The detour distance between v_2 and v_{n-1} is $n - 2$ ...……… The detour distance between v_{n-2} and v_{n-1} is 1. In the same way if we find all the detour distances between every pair of vertices of G, we get (n−1)-times 1, (n−2)-times 2, (n−3)-times 3, (n−4)-times 4, (n−5)-times 5,…….,3 times (n−3), 2 times (n−2) and 1 time (n−1). Find the median of those nC_2 possible detour distances by arranging it in an ascending order. If nC_2 is odd, then $\frac{nC_2}{2}$th term will be the halfway distance. If nC_2 is even then the average of $\frac{nC_2}{2}$ and $\frac{nC_2}{2}$+1 th term will be the halfway distance. Since in the set of possible detour distances only one value is n−1, $H(u, v)$ is always less than $n - 1$.

2. Halfway Set

2.1.General Bound for halfway set: hn(G) ranges between 2 and n with sharpness in both upper and lower bound. Sharpness in lower bound is for complete graph and cycle C_n where n = 3,4,5 and in upper bound it is for path P_3.

2.2.Observation

1. For any complete graph G, $hn(G)$ = 2.
2. Let G be the wheel graph. Then $hn(G)$ = 2.

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3. For the path graph G the halfway number $hn(G)$ can be found by the following procedure.

Step 1:

Let $V(G) = \{ v_1, v_2, \dots, v_n\}$ be the set of all vertices of G such that v_1 and v_n are the pendant vertices. Let S be the halfway set of G. Starting with v_1 find the detour distance of every pair of vertices of G, the median of which gives the halfway distance $H(u,v)$.

Step 2:

Take S = $\{ v_1\}$. Starting with v_1 the next vertex in the set S will be $v_{1+H(u,v)}$. That is S = $\{ v_1, v_{1+H(u,v)}\}$. When $1 + H(u,v)$ exceeds n where n is the order of G, go to step 4. Otherwise go to step 3.

Step3:

When $1 + H(u,v)$ is less than n, add $H(u,v)$ to $1 + H(u,v)$. That is, the next vertex in the set S will be $v_{1+H(u,v)+H(u,v)}$. Therefore S = $\{ v_1, v_{1+H(u,v)}, v_{1+H(u,v)+H(u,v)}\}$. If 1 +$H(u,v) + H(u,v)$ exceeds n, stop and go to step 4. Otherwise repeat step 3 until v_n appears in the set S.

Step 4:

If the value of 1+$H(u,v)$+ $H(u,v) + H(u,v)\dots$ exceeds n, then replace 1 +$H(u,v) + H(u,v) + H(u,v)\dots$ by n and stop….Then we get $S = \{ v_1, v_{1+H(u,v)}, v_{1+H(u,v)+H(u,v)}, \dots, v_n\}$. Thus $hn(G) = |S|$.

Example:

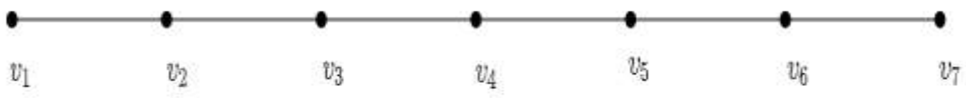


Fig.2

Step 1:

Find the detour distance of every pair of vertices of the above graph. Let K be the set of all detour distances. Then K={1,2,3,4,5,6,1,2,3,4,5,1,2,3,4,1,2,3,1,2,1}. Halfway distance is nothing but median of K which is 2.

Step 2:

Take $S =\{v_1\}$. The next vertex in S is $\{v_{1+H(u,v)}\} = \{v_{1+2}\} =\{v_3\}$. Therefore $S =\{v_1, v_3\}$. Now $1 + H(u,v) = 3 < n$, go to step 3.

Step 3:

Add $H(u,v)$ to $1 + H(u,v)$. Therefore the next vertex is v_5. Thus S=$\{v_1, v_3, v_5\}$. Now $1 + H(u,v) + H(u,v) = 5 < n$. Repeat step 3. That is add $H(u,v)$ to $1 + H(u,v) + H(u,v)$ which gives S =$\{v_1, v_3, v_5, v_7\}$. Now the vertex v_7 appears in the set S. Therefore STOP! Thus we get S=$\{v_1, v_3, v_5, v_7\}$ and hence $hn(G) = 4$.

Theorem 2.3. The halfway number of the cycle graph is $hn(G) = \begin{cases} 2 & n = 3,4 \\ 3 & n \geq 5 \end{cases}$

Proof. We are going to prove this theorem by considering the following cases.

Case (i): When $n = 3$

In this case $H(u,v) = 2$. Therefore we get $hn(G) = 2$.

Case (ii): When $n = 4$

Since the degree of each vertex is 2 and $H(u,v) = 3$, the halfway set will contain only 2 vertices satisfying the basic requirements to be a halfway set. Thus $hn(G) = 2$.

Case (iii): When $n \geq 5$

Let $\{v_1, v_2, \dots, v_n\}$ be the set of all vertices of the cycle graph. Let K be the set of all detour distances of every pair of vertices of G. The halfway distance $H(u,v)$ can be determined by taking the median of K.

Let S be the halfway set. Suppose $H(u,v) = r$ for some positive integer $1 \leq r \leq n$. Consider v_i, v_j for some $1 \leq i, j \leq n$. The path $v_i - v_j$ satisfies the halfway distance.

Now $H(u,v) = r$ and $|V(G)| = n$ implies there exists some other vertices which is not contained in S. Since G is a cycle and $\delta = 2$ we can find some other halfway path $v_i - v_t$ which contains the leftover vertices of G. That is, S =$\{v_i, v_j, v_t\}$ for some $1 \leq t \leq n$ and hence $hn(G) = 3$.

Example. In the below figure K={6,5,4,4,5,6,6,5,4,4,5,6,5,4,4,6,5,4,6,5,6}. We have $H(u,v)$= Med(K)= {4,4,4,4,4,4,4,5,5,5,5,5,5,5,6,6,6,6,6,6,6}. Consider $v_1 - v_6$. The halfway path between $v_1$and v_6 contains the vertices v_2, v_3, v_4, v_5. But the vertex v_7 does not belong to the $v_1 - v_6$ halfway path. Therefore we need to consider another path ie.,$v_1 - v_3$.This path contains the vertex v_7. Thus S=$\{v_1, v_6, v_3\}$forms a minimum halfway set and hence $hn(G) = 3$.

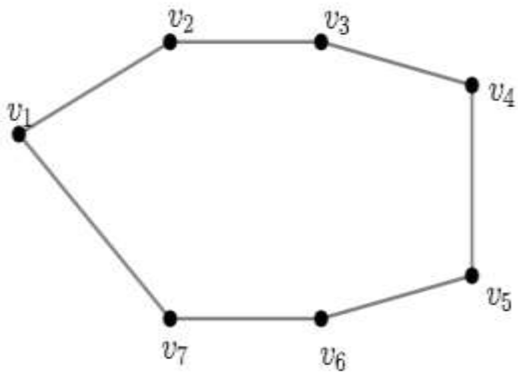


Fig.3

Observation 2.4. The vertices of degree one belongs to the halfway set.

Proof. Let S be the halfway set of G. Suppose the vertices of degree one does not belongs to the halfway set. By the definition of halfway set all the vertices of G belong to some halfway path of vertices of S. Our assumption shows that S is not a halfway set which is a contradiction. Thus the vertices of degree one belongs to the halfway set.

Observation 2.5. Support vertices need not belongs to the halfway set.
Example 1.

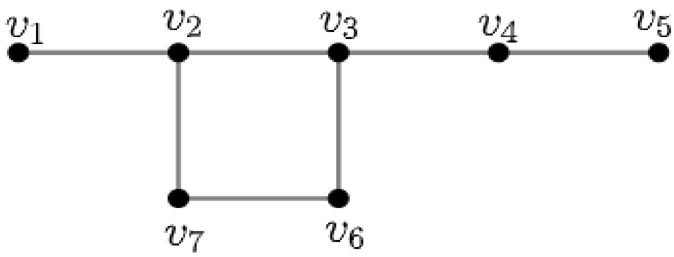


Fig.4
Here $H(u, v)$ =med($\{1,1,1,1,2,2,2,3,3,3,3,3,4,4,4,4,5,5,5,6\}$) and hence the halfway distance is $H(u, v) = 3$. The support vertices are v_2 and v_4. But the halfway set is S=$\{v_1, v_6, v_5\}$ which does not contain the support vertices.
Example 2. In this example, the halfway distance is $H(u, v) = 4$ and hence $S = \{v_1, v_2, v_4, v_5, v_6, v_7, v_8\}$ is the halfway set. Here the set S contain all the support vertices.

Thus from examples 1 and 2 we see that support vertices need not belongs to the halfway set.

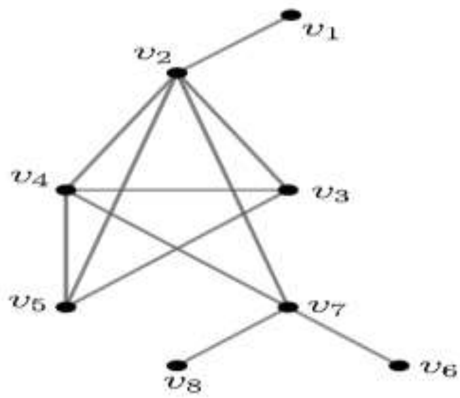
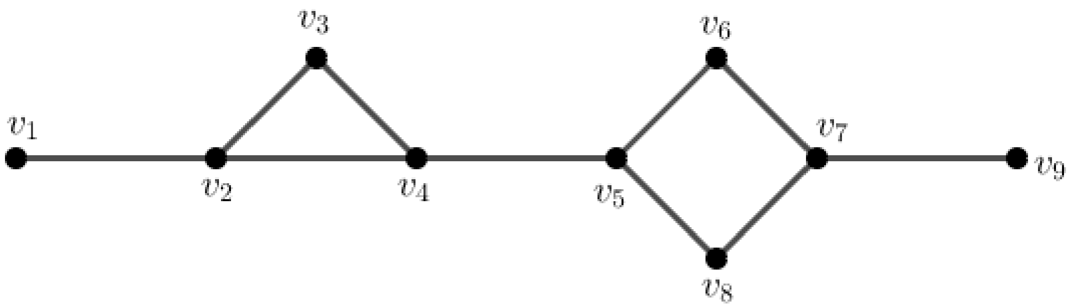


Fig.5

Result 2.6. The halfway distance is less than or equal to the detour distance, that is $H(u,v) \leq D(u,v)$.
Proof. Since halfway distance is the median of the set of all possible detour distances between vertices of G, the result follows.
Result 2.7. Let L be the set of all pendant vertices of G and S be the halfway set. When we consider trees or graphs with pendant vertices the hn -set contains all the pendant vertices together with some more vertices. Hence L⊆ S.
Result 2.8. For any connected graph G, $dn \leq hn$.

Theorem 2.9. If T is a tree with k pendant vertices, then $hn(G) \geq k$.
Proof. Let T be a tree of order n with k pendant vertices. By theorem, all the pendant vertices belongs to the halfway set. Therefore the halfway set of the tree must contains the k pendant vertices. Thus $hn(G) \geq k$.

Observation 2.10. Every cut vertex need not belongs to a halfway set.
Example: Let $\{v_1, v_2, v_3, \dots, v_9\}$ be the vertices of G. Let $\{v_2, v_4, v_5, v_7\}$ be the cut vertices of G. We get the median of the possible detour distances is 4. That is $H(u,v) = 4$. The halfway set of G is S=$\{v_1, v_4, v_5, v_9\}$. It is clear that the vertices v_2 and v_7 does not belong to S. Thus not all the cut vertices belong to S.



G
Observation 2. Let G be the Peterson Graph. Then $hn(G) = 2$.
Observation 3. If $H(u,v) = D(u,v)$ then $dn = hn$.
3.Characterisation for $hn(G) = 2$

Theorem 3.1. Let G be a connected graph such that $H(u,v) = 1$. Then $hn(G) = 2$ iff G≅ P_2
Proof. Assume that $hn(G) = 2$. To prove G≅ P_2. Since $H(u,v) = 1$ and $hn(G) = 2$, it is clear that G≅ P_2. Conversely, assume G≅ P_2. Order of P_2 is 2 implies $hn(G) = 2$.

Theorem 3.2. Let G be a connected graph of order n with no pendant vertices. If $H(u,v) = n - 1$, then $hn(G) = 2$, $u, v \in V(G)$.
Proof. Let K be the set of all possible detour distances between pair of vertices of G. $H(u,v)$ is the median of K. Let us assume that $H(u,v) = n - 1$. Then the median of the possible detour distances is $n - 1$. That is there exists a detour path of length n-1 between u and v which contains all the vertices of G for some u,v ∈ $V(G)$. Thus the set S = {u,v} forms a minimum halfway set, for some u,v ∈ $V(G)$. Therefore $hn(G) = 2$.

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Observation 3.3. The converse of the above theorem need not be true. Because for the complete bipartite graph $hn(G) = 2$, but $H(u, v) \neq n - 1$.

4.Halfway Number of Special Graphs:

Definition 4.1. The *double star* S(n,m), where n$\geq m \geq 0$, is the graph consisting of the union of the two stars $K_{1,n}$ and $K_{1,m}$ together with a line joining their centers.

Theorem 4.2. Let G be a connected double star of order $n + m + 2$. Then $hn(G) = n + m$.
Proof. Let $V(G) = \{v_1, v_2, v_3, \dots, v_{n+m}, v_{n+m+1}, v_{n+m+2}\}$ be the vertices of the double star G. In G, the vertices v_{n+1} and v_{n+2} are of degree n+1 and m+1 respectively. All the other vertices are of degree one. Now by theorem, that pendant vertices belongs to the halfway set, we have $\{v_1, v_2, \dots, v_n, v_{n+3}, \dots, v_{n+m+2}\}$ belongs to the halfway set. Let S be the halfway set. The vertices v_{n+1} and v_{n+2} lie in any one of the halfway path of vertices of S. Thus we get $hn(G) = |S| = n + m$.

Example: Let K= {1,2,2,2,3,3,3,3,1,1,1,2,2,2,2,2,2,2,3,3,3,3,2,3,3,3,3,1,1,1,1,2,2,2,2,2,2} be the set of all possible detour distances of the double star graph $S(3,4)$. $H(u, v) =$ Median of K.Therefore $H(u, v)$ =median(1,2,2,2,3,3,3,3,1,1,1,2,2,2,2,2,2,2,3,3,3,3,2,3,3,3,3,1,1,1,1,2,2,2,2,2,2) = 2. Thus S = $\{v_1, v_3, v_4, v_6, v_7, v_8, v_9\}$ which implies $hn(G) = 7$.

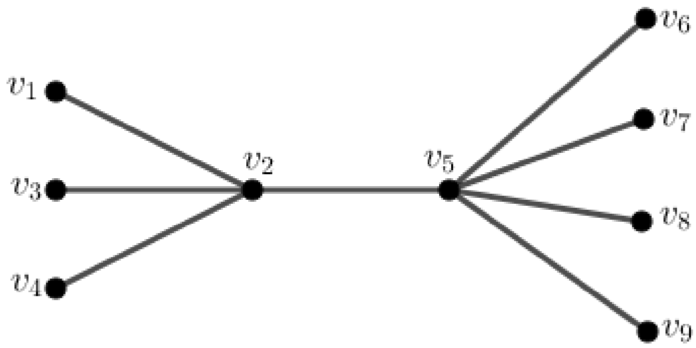

Fig. Double star $S(3,4)$

Definition 4.3. The friendship graph F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex, which becomes a universal vertex for the graph.
Theorem 4.4. Let G be the friendship graph of order 2n+1 where $n \geq 3$. Then $hn(G) = n$.
Proof. Let the vertices of G be $\{v_1, v_2, \dots, v_{2n+1}\}$. Let us find the possible detour distances between pair of vertices of G and find the halfway distance of G. For G, each vertex from the n copy of C_3 will form the halfway set of G. Let S = $\{ v_1, v_3, \dots, v_{2n-1}\}$. Then $|S| = n$. Thus $hn(G) = n$.

Definition 4.5. The n- pan graph is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge.
Theorem 4.6. Let G be a n- Pan Graph. Then $hn(G) = 3$ where $n \geq 7$.
Proof. The n-Pan Graph contains n-1 vertices of degree 2, one vertex of degree 3 and a vertex of degree 1. By theorem pendant vertices belongs to the detour distances of vertices, we get the halfway distance. Thus we get $S = \{v_i, v_j, v_{n+1}\}$ forms the halfway set because the halfway distance between vertices of S contains all the vertices of G. Therefore $hn(G) = 3, n \geq 7$.

5. Halfway Number of Corona Graphs

In this section we go through the halfway number of some corona product of graphs.
Definition: 5.1 The ***corona product*** $G\ o\ H$of two graphs G and H is obtained by taking one copy of G and |V(G)| copies of H and by joining each vertex of the i^{th} copy of H to the i^{th} vertex of G where $1 \leq i \leq |V(G)|$.
Theorem 5.2. If G is corona of P_n and K_1 , then hn(G)=n+$\left\lfloor\frac{n}{H(u,v)}\right\rfloor$, $n \geq 2$.
Proof. Let G be the corona of the path graph P_n and K_1. Then G has 2n vertices so that $V(G)$ =$\{v_1, v_2, \dots v_{2n}\}$. To find the halfway number, we need to find the set of all possible detour distances of the vertices in V(G).

Let K be such set and S be the halfway set of G. H(u,v) is the median of K. A=$\{v_2, v_4, v_6, v_8, \dots, v_{2n}\}$ be the set of all pendant vertices of G. By theorem pendant vertices belongs to $hn(G)$. The elements of A belongs to the halfway set S, implies A$\subseteq S$. $H(u,v)$ is the halfway distance and $|V(G)| = n$, gives $\frac{n}{H(u,v)}$ elements will also belongs to the halfway set.

$\Rightarrow\ |S| = |A| + \left\lfloor \frac{n}{H(u,)} \right\rfloor$

$\Rightarrow\ hn(G) = n + \left\lfloor \frac{n}{H(u,v)} \right\rfloor$

Theorem 5.3. Let G be the connected graph $K_{1,n} \odot K_1$ of order $2(n+1)$. Then $hn(G) = \begin{cases} n+2 & if\ H(u,v) = 2 \\ n+1 & if\ \text{H(u,v)} = 3 \end{cases}$

Proof. Let S be the halfway set. Let $\{v_1, v_2, \dots, v_{2n+2}\}$ be the vertices of G where $\{v_1, v_4, v_6, v_8, \dots, v_{2n+2}\}$ are the pendant vertices of G. By theorem the vertices of degree one belongs to halfway set. Then $\{v_1, v_4, \dots, v_{2n+2}\}$ belongs to S.

i) Since the halfway distance $H(u,v) = 2$ the halfway distance from v_1 to every other vertices of S should be atmost 2. But detour distance from v_1 to all other vertices of S is 3. Therefore S contains one more vertex v_2. Hence S= $\{v_1, v_4, v_6, v_8, \dots, v_{2n+2}\} \cup \{v_2\}$ implies $|S| = n+1+1 = n+2$. Thus $hn(G) = n+2$.

ii) Since the Halfway distance $H(u,v) = 3$ the halfway distances from v_1 to every other vertex of S should be atmost 3. Consider $v_1 - v_4$ path. Halfway distance between them is 3. In the same way halfway distance from v_1 to all other vertices $v_4, v_6, v_8, \dots, v_{2n+2}$ is 3. Therefore $S = \{v_1, v_4, v_6, v_8, \dots, v_{2n+2}\}$ forms a halfway set. Thus $|S| = n+1$ implies $hn(G) = n+1$.

Theorem 5.4. Let G be the connected graph $P_n \odot \overline{K}_2$. Then $hn(G) = 2n$.

Proof. Let $\{v_1, v_2, \dots, v_{3n}\}$ be the vertices of the corona graph $P_n \odot \overline{K}_2$. The vertices $\{v_2, v_3, v_5, v_6, \dots, v_{3n-1}, v_{3n}\}$ are of degree one. By theorem, pendant vertices belongs to the halfway set. Hence $|S| = 2n \Rightarrow hn(G) = 2n$.

6.Conclusion

In this paper we have introduced the concept of halfway distance. Also we have found some of the relation between halfway distance and detour distance. It's interesting while looking into the opening results that $H(u,v) \leq D(u,v)$ while $dn \leq hn$. Also halfway number of some special class of graphs and corona graphs have been determined.

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