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Analysis Of Queue Network Model With Three Parallel Servers Connected In Series

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ABSTRACT:

The present paper analyses a queue network model comprising of parallel subsystem commonly linked with two different service channels in series. The arrival follows the poison distribution laws. The various queue parameters have been derived using generating function technique and laws of calculus in steady state condition. Numerical illustration provided validates the model in better way. The proposed model finds its applications in various fields like administrative setup, shopping malls, hospitals, banking sector and many similar real life situation.

KEYWORDS: Generating function technique, queue length, average waiting time, parallel server, serial server.

1.INTRODUCTION:

Queuing theory deals with the study of customers waiting in line and examines its various components i.e. arrival process, number of servers and no. of customers that might be cars, people, data packets or anything else. Formation of queue is a part of our routine life. Various queuing models have been formed to provide better services to the customer. One of the earliest Study in queuing theory was procured by A.K. Erlang (1909) to examine the waiting time in telephone services. Jacksons (1954) deliberate the way of behaving of a queue system carrying phase type facility. Suzuki (1963) examines the action of arrangements when two queues are in series. Shaler Stidham Jr. et.al. (1993) developed a model based on Markov decision theory for the optimal control of network of queues. Huimin Xiao et.al. (2010) discussed application of queuing theory in banking system. Deepak Gupta et.al. (2012) deals with the linkage between queue network to obtain minimize elapsed time and mean queue length i.e. two fold objective. AVS Suhasini et.al. (2013) analyse a time dependent queuing model containing parallel and series configuration. Deepak Gupta (2013) et.al. considered a model consisted of biserial server linked to a common server in fuzzy environment. Md.Al- Amin Molla (2017) develop a single service channel waiting model. Harminder Singh et.al. (2019) analyse a network queue model be made up of parallel service channels linked with a common server. Deepak Gupta et.al. (2021) studied steady state behaviour of a complex model containing biserial and parallel service channel connected in series with single server. Vandana Saini (2021) et.al. discussed a complex model containing three subsystems. Deepak Gupta (2022) et.al. developed a queue network model with fixed batch size. Deepak Gupta (2022) et.al. studied a feedback queue model comprised of biserial and serial service channel. The Present manuscript explores a queuing system, where a single server is connected in series to three parallel servers and these servers are further associated to another single server for the completion of service.

2.PRACTICAL APPLICATION:

The proposed model finds its applications in Hospitals, Banking System and Manufacturing concern etc. Let us consider the situation of multi specialist Hospital. Suppose that there are three sections, first is registration section, second section is for given therapy treatment that further contains three subsections, one is of Physiotherapy, second is of Naturopathy and third is of Acupressure therapy and last section is for paying bill. The Patient come in Hospital firstly move to registration section and after getting registered the patient is allowed to move either one of the three sections, after taking treatment one of the therapy according to their need, Patient clears the bill and then leave the system.

3. NOTATIONS

The notation used in the analysis of queuing model are as follows:

Number of customers	c_1	c_2	c_3	c_4	c_5
Service Channels	s_1	s_2	s_3	s_4	s_5
Arrival rate	λ_1				
Service rate	μ_1	μ_2	μ_3	μ_4	μ_5
Probability of Customer switching between service channel	$S_1 \rightarrow S_2$ α_{12}	$S_1 \rightarrow S_2$ α_{13}	$S_1 \rightarrow S_2$ α_{14}		

4. DESCRIPTION OF THE MODEL

The model under consideration in the queuing system consist of three service channels s_1 , s'_1 *and* s_5.Further the service channel s'_1 contains three parallel subservice channel s_2, s_3 and s_4. Let there are c_1, c_2, c_3, c_4 *and* c_5 number of customers in front of the service channel s_1, s_2 , s_3, s_4 , s_5 respectively. The service channel s'_1 is commonly connected in series with two service channels s_1, s_5. Let λ_1 is arrival rate at service channel s_1. Let μ_1, μ_2, μ_3, μ_4 and μ_5 is average service rate at service channel s_1, s_2, s_3, s_4, s_5 respectively. The customer come in the front of service channel s_1,formed a queue and will take their service and after taking service from s_1 the customer will allowed to move either one of the three subservice channels s_2, s_3, s_4 with probability α_{12} , α_{13} and α_{14} such that $\alpha_{12} + \alpha_{13} + \alpha_{14} = 1$ and then customer will move to the service channel s_5 and after getting service leave the system.

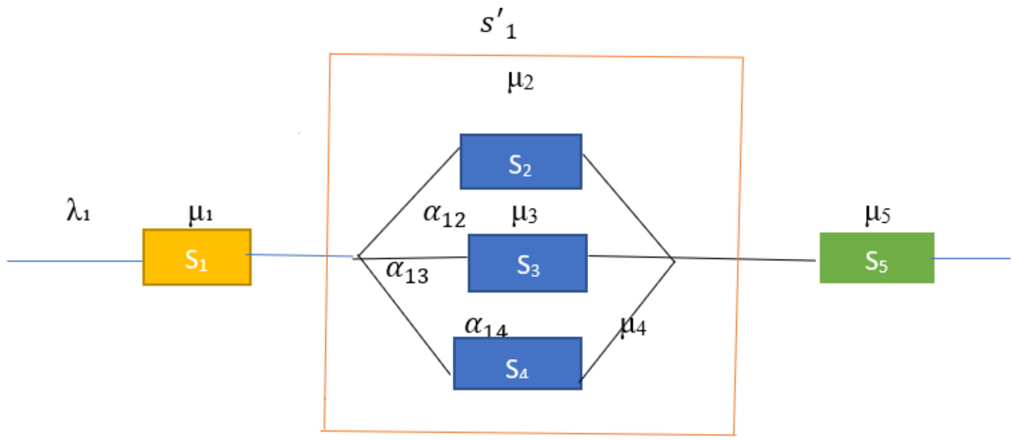

Figure 1.Queuing network model

FORMULATION OF DIFFERENTIAL DIFFERENCE EQUATIONS:

Let P_{c_1,c_2,c_3,c_4,c_5} represents the probability that there are c_1, c_2, c_3, c_4, c_5 number of customers in front of s_1, s_2, s_3, s_4, s_5 service channel waiting to take its service respectively.

The differential difference equations in transient state are as follows:

For $c_1>0$, $c_2>0$, c_3, c_4, $c_5>0$

$$P'_{c_1,c_2,c_3,c_4,c_5}(t) = -(\lambda_1+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5) P_{c_1,c_2,c_3,c_4,c_5}(t)+\lambda_1 P_{c_1-1,c_2,c_3,c_4,c_5}(t)+\mu_1\alpha_{12}P_{c_1+1,c_2-1,c_3,c_4,c_5}(t)+\mu_1\alpha_{13}P_{c_1+1,c_2,c_3-1,c_4,c_5}(t)+\mu_1\alpha_{14}P_{c_1+1,c_2,c_3,c_4-1,c_5}(t)+\mu_2 P_{c_1,c_2+1,c_3,c_4,c_5-1}(t)+\mu_3 P_{c_1,c_2,c_3+1,c_4,c_5-1}(t)+\mu_4 P_{c_1,c_2,c_3,c_4+1,c_5-1}(t)+\mu_5 P_{c_1,c_2,c_3,c_4,c_5+1}(t)$$

The differential difference equations in steady state are as follows:

For $c_1>0$, $c_2>0$, c_3, c_4, $c_5>0$

$$(\lambda_1+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5) P_{c_1,c_2,c_3,c_4,c_5}(t)=\lambda_1 P_{c_1-1,c_2,c_3,c_4,c_5}(t)+\mu_1\alpha_{12}P_{c_1+1,c_2-1,c_3,c_4,c_5}(t)+\mu_1\alpha_{13}P_{c_1+1,c_2,c_3-1,c_4,c_5}(t)+\mu_1\alpha_{14}P_{c_1+1,c_2,c_3,c_4-1,c_5}(t)+\mu_2 P_{c_1,c_2+1,c_3,c_4,c_5-1}(t)+\mu_3 P_{c_1,c_2,c_3+1,c_4,c_5-1}(t)+\mu_4 P_{c_1,c_2,c_3,c_4+1,c_5-1}(t)+\mu_5 P_{c_1,c_2,c_3,c_4,c_5+1}(t)$$

By considering all the possible combination of different values of c_1, c_2, c_3, c_4, c_5 , 32 equations are obtained.

For $c_1>0$, $c_2>0$, c_3, c_4, $c_5>0$

$$(\lambda_1+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P_{c_1,c_2,c_3,c_4,c_5}(t)=\lambda_1 P_{c_1-1,c_2,c_3,c_4,c_5}(t)+\mu_1\alpha_{12}P_{c_1+1,c_2-1,c_3,c_4,c_5}(t)+\mu_1\alpha_{13}P_{c_1+1,c_2,c_3-1,c_4,c_5}(t)+\mu_1\alpha_{14}P_{c_1+1,c_2,c_3,c_4-1,c_5}(t)+\mu_2 P_{c_1,c_2+1,c_3,c_4,c_5-1}(t)+\mu_3 P_{c_1,c_2,c_3+1,c_4,c_5-1}(t)+\mu_4 P_{c_1,c_2,c_3,c_4+1,c_5-1}(t)+\mu_5 P_{c_1,c_2,c_3,c_4,c_5+1}(t) \quad (1)$$

For $c_1=0$, $c_2, c_3, c_4, c_5>0$

$$(\lambda_1+\mu_2+\mu_3+\mu_4+\mu_5) P_{0,c_2,c_3,c_4,c_5} = \mu_1\alpha_{12}P_{1,c_2-1,c_3,c_4,c_5} + \mu_1\alpha_{13}P_{1,c_2,c_3-1,c_4,c_5} + \mu_1\alpha_{14}P_{1,c_2,c_3,c_4-1,c_5}+\mu_2 P_{0,c_2+1,c_3,c_4,c_5-1} + \mu_3 P_{0,c_2,c_3+1,c_4,c_5-1} + \mu_4 P_{0,c_2,c_3,c_4+1,c_5-1} + \mu_5 P_{0,c_2,c_3,c_4,c_5+1} \quad (2)$$

For $c_1>0$, $c_2=0$, $c_3, c_4, c_5>0$

$$(\lambda_1+\mu_1+\mu_3+\mu_4+\mu_5) P_{c_1,0,c_3,c_4,c_5} = \lambda_1 P_{c_1,0,c_3,c_4,c_5}+ \mu_1\alpha_{13}P_{c_1+1,0,c_3-1,c_4,c_5} + \mu_1\alpha_{14}P_{c_1+1,0,c_3,c_4-1,c_5}+\mu_2 P_{c_1,1,c_3,c_4,c_5-1} +\mu_3 P_{c_1,0,c_3+1,c_4,c_5-1}+\mu_4 P_{c_1,0,c_3,c_4+1,c_5-1} + \mu_5 P_{c_1,0,c_3,c_4,c_5+1} \quad (3)$$

For $c_1=0$, $c_2=0$, $c_3, c_4, c_5>0$

$$(\lambda_1+\mu_3+\mu_4+\mu_5) P_{0,0,c_3,c_4,c_5} = \mu_1\alpha_{13}P_{1,0,c_3-1,c_4,c_5} + \mu_1\alpha_{14}P_{1,0,c_3,c_4-1,c_5}+\mu_2 P_{0,1,c_3+1,c_4,c_5-1}+\mu_3 P_{0,0,c_3+1,c_4,c_5-1}+\mu_4 P_{0,0,c_3,c_4+1,c_5-1}+\mu_5 P_{0,0,c_3,c_4,c_5+1} \quad (4)$$

For $c_1>0$, $c_2>0$, $c_3=0$, c_4, $c_5>0$

$$(\lambda_1+\mu_1+\mu_2+\mu_4+\mu_5) P_{c_1, c_2, 0, c_4, c_5} = \lambda_1 P_{c_1, c_2, 0, c_4, c_5} + \mu_1\alpha_{12} P_{c_1+1, c_2-1, 0, c_4, c_5} + \mu_1\alpha_{14} P_{c_1+1, c_2, 0, c_4-1, c_5} + \mu_2 P_{c_1, c_2+1, 1, c_4, c_5-1} + \mu_3 P_{c_1, c_2, 1, c_4, c_5-1} + \mu_4 P_{c_1, c_2, 0, c_4+1, c_5-1} + \mu_5 P_{c_1, c_2, 0, c_4, c_5+1} \quad (5)$$

For $c_1=0$, $c_2>0$, $c_3=0$ c_4, $c_5>0$

$$(\lambda_1+\mu_2+\mu_4+\mu_5) P_{0, c_2, 0, c_4, c_5} = \mu_1\alpha_{12} P_{1, c_2-1, 0, c_4, c_5} + \mu_1\alpha_{14} P_{1, c_2, 0, c_4-1, c_5} + \mu_2 P_{0, c_2+1, 1, c_4, c_5-1} + \mu_3 P_{0, c_2, 1, c_4, c_5-1} + \mu_4 P_{0, c_2, 0, c_4+1, c_5-1} + \mu_5 P_{0, c_2, 0, c_4, c_5+1} \quad (6)$$

For $c_1>0$, $c_2>0$, c_3, $c_4=0$ $c_5>0$

$$(\lambda_1+\mu_1+\mu_2+\mu_3+\mu_5) P_{c_1, c_2, c_3, 0, c_5} = \lambda_1 P_{c_1, c_2, c_3, 0, c_5} + \mu_1\alpha_{12} P_{c_1+1, c_2-1, c_3, 0, c_5} + \mu_1\alpha_{13} P_{c_1+1, c_2, c_3-1, 0, c_5} + \mu_2 P_{c_1, c_2+1, c_3+1, 0, c_5-1} + \mu_3 P_{c_1, c_2, c_3+1, 0, c_5-1} + \mu_4 P_{c_1, c_2, c_3, 1, c_5-1} + \mu_5 P_{c_1, c_2, c_3, 0, c_5+1} \quad (7)$$

For $c_1=0$, $c_2>0$, c_3, $c_4=0$ $c_5>0$

$$(\lambda_1+\mu_2+\mu_3+\mu_5) P_{0, c_2, c_3, 0, c_5} = \mu_1\alpha_{12} P_{1, c_2-1, c_3, 0, c_5} + \mu_1\alpha_{13} P_{1, c_2, c_3-1, 0, c_5} + \mu_2 P_{0, c_2+1, c_3+1, 0, c_5-1} + \mu_3 P_{0, c_2, c_3+1, 0, c_5-1} + \mu_4 P_{0, c_2, c_3, 1, c_5-1} + \mu_5 P_{0, c_2, c_3, 0, c_5+1} \quad (8)$$

For $c_1>0$, $c_2>0$, c_3, c_4, $c_5=0$

$$(\lambda_1+\mu_1+\mu_2+\mu_3+\mu_4) P_{c_1, c_2, c_3, c_4, 0} = \lambda_1 P_{c_1, c_2, c_3, c_4, 0} + \mu_1\alpha_{12} P_{c_1+1, c_2-1, c_3, c_4, 0} + \mu_1\alpha_{13} P_{c_1+1, c_2, c_3-1, c_4, 0} + \mu_1\alpha_{14} P_{c_1+1, c_2, c_3, c_4-1, 0} + \mu_5 P_{0, c_2, c_3, 0, 1} \quad (9)$$

For $c_1=0$, $c_2>0$, c_3, c_4, $c_5=0$

$$(\lambda_1+\mu_2+\mu_3+\mu_4) P_{0, c_2, c_3, c_4, 0} = \mu_1\alpha_{12} P_{1, c_2-1, c_3, c_4, 0} + \mu_1\alpha_{13} P_{1, c_2, c_3-1, c_4, 0} + \mu_1\alpha_{14} P_{1, c_2, c_3, c_4-1, 0} + \mu_5 P_{0, c_2, c_3, 0, 1} \quad (10)$$

For $c_1>0$, $c_2 = 0$, $c_3 = 0$, c_4, $c_5>0$

$$(\lambda_1+\mu_1+\mu_4+\mu_5) P_{c_1, 0, 0, c_4, c_5} = \lambda_1 P_{c_1, 0, 0, c_4, c_5} + \mu_1\alpha_{14} P_{c_1+1, 0, 0, c_4-1, c_5} + \mu_2 P_{c_1, 1, 1, c_4, c_5-1} + \mu_3 P_{c_1, 0, 1, c_4, c_5-1} + \mu_4 P_{c_1, 0, 0, c_4+1, c_5-1} + \mu_5 P_{c_1, 0, 0, c_4, c_5+1} \quad (11)$$

For $c_1=0$, $c_2 = 0$, $c_3 = 0$, c_4, $c_5>0$

$$(\lambda_1+\mu_1+\mu_4+\mu_5) P_{0, 0, 0, c_4, c_5} = \mu_1\alpha_{14} P_{1, 0, 0, c_4-1, c_5} + \mu_2 P_{0, 1, 1, c_4, c_5-1} + \mu_3 P_{0, 0, 1, c_4, c_5-1} + \mu_4 P_{0, 0, 0, c_4+1, c_5-1} + \mu_5 P_{0, 0, 0, c_4, c_5+1} \quad (12)$$

For $c_1>b_1$, $c_2 = 0$, $c_3>0$, $c_4 = 0$, $c_5>0$

$$(\lambda_1+\mu_1+\mu_3+\mu_5) P_{c_1, 0, c_3, 0, c_5} = \lambda_1 P_{c_1-b_1, 0, c_3, 0, c_5} + \mu_1\alpha_{13} P_{c_1+1, 0, c_3-1, 0, c_5} + \mu_2 P_{c_1, 1, c_3+1, 0, c_5-1} + \mu_3 P_{c_1, 0, c_3+1, 0, c_5-1} + \mu_4 P_{c_1, 0, c_3, 1, c_5-1} + \mu_5 P_{c_1, 0, c_3, 0, c_5+1} \quad (13)$$

For $c_1=0$, $c_2 = 0$, $c_3>0$, $c_4 = 0$, $c_5>0$

$$(\lambda_1+\mu_3+\mu_5) P_{0, 0, c_3, 0, c_5} = \mu_1\alpha_{13} P_{1, 0, c_3-1, 0, c_5} + \mu_2 P_{0, 1, c_3+1, 0, c_5-1} + \mu_3 P_{0, 0, c_3+1, 0, c_5-1} + \mu_4 P_{0, 0, c_3, 1, c_5-1} + \mu_5 P_{0, 0, c_3, 0, c_5+1} \quad (14)$$

For $c_1>0$, $c_2 = 0$, $c_3, c_4>0$, $c_5=0$

$$(\lambda_1+\mu_1+\mu_3+\mu_4) P_{c_1, 0, c_3, c_4, 0} = \lambda_1 P_{c_1, 0, c_3, c_4, 0} + \mu_1\alpha_{13} P_{c_1+1, 0, c_3-1, c_4, 0} + \mu_1\alpha_{14} P_{c_1+1, 0, c_3, c_4-1, 0} + \mu_5 P_{0, c_2, c_3, 0, 1} \quad (15)$$

For $c_1=0$, $c_2 = 0$, $c_3, c_4>0$, $c_5=0$

$$(\lambda_1+\mu_3+\mu_4) P_{0, 0, c_3, c_4, 0} = \mu_1\alpha_{13} P_{1, 0, c_3-1, c_4, 0} + \mu_1\alpha_{14} P_{1, 0, c_3, c_4-1, 0} + \mu_5 P_{0, c_2, c_3, 0, 1} \quad (16)$$

For $c_1>0$, $c_2 > 0$, $c_3 = 0$, $c_4 = 0$, $c_5>0$

$$(\lambda_1+\mu_1+\mu_2+\mu_5) P_{c_1, c_2, 0, 0, c_5} = \lambda_1 P_{c_1, c_2, 0, 0, c_5} + \mu_1\alpha_{12} P_{c_1+1, c_2-1, 0, 0, c_5} + \mu_2 P_{c_1, c_2+1, 1, 0, c_5-1} + \mu_3 P_{c_1, c_2, 1, 0, c_5-1} + \mu_4 P_{c_1, c_2, 0, 1, c_5-1} + \mu_5 P_{c_1, c_2, 0, 0, c_5+1} \quad (17)$$

For $c_1=0$, $c_2 > 0$, $c_3 = 0$, $c_4 = 0$, $c_5>0$

$$(\lambda_1+\mu_2+\mu_5) P_{0, c_2, 0, 0, c_5} = \mu_1\alpha_{12} P_{1, c_2-1, 0, 0, c_5} + \mu_2 P_{0, c_2+1, 1, 0, c_5-1} + \mu_3 P_{0, c_2, 1, 0, c_5-1} + \mu_4 P_{0, c_2, 0, 1, c_5-1} + \mu_5 P_{0, c_2, 0, 0, c_5+1} \quad (18)$$

For $c>0$, $c_2 > 0$, $c_3 = 0$, $c_4>0$, $c_5=0$

$$(\lambda_1+\mu_1+\mu_2+\mu_4) P_{c_1, c_2, 0, c_4, 0} = \lambda_1 P_{c_1, c_2, 0, c_4, 0} + \mu_1\alpha_{12} P_{c_1+1, c_2-1, 0, c_4, 0} + \mu_1\alpha_{14} P_{c_1+1, c_2, 0, c_4-1, 0} + \mu_5 P_{c_1, c_2, c_3, c_4, 1} \quad (19)$$

For $c_1=0$, $c_2 > 0$, $c_3 = 0$, $c_4>0$, $c_5=0$

$$(\lambda_1+\mu_1+\mu_2+\mu_4) P_{c_1, c_2, 0, c_4, 0} = \mu_1\alpha_{12} P_{1, c_2-1, 0, c_4, 0} + \mu_1\alpha_{14} P_{1, c_2, 0, c_4-1, 0} + \mu_5 P_{0, c_2, c_3, c_4, 1} \quad (20)$$

For $c_1>0$, $c_2>0$, $c_3 > 0$, $c_4=0$, $c_5=0$

$$(\lambda_1+\mu_1+\mu_2+\mu_3) P_{c_1, c_2, c_3, 0, 0} = \lambda_1 P_{c_1, c_2, c_3, 0, 0} + \mu_1\alpha_{12} P_{c_1+1, c_2-1, c_3, 0, 0} + \mu_1\alpha_{13} P_{c_1+1, c_2, c_3-1, 0, 0} + \mu_5 P_{c_1, c_2, c_3, 0, 1} \quad (21)$$

For $c_1=0$, $c_2>0$, $c_3 > 0$, $c_4=0$, $c_5=0$

$(\lambda_1+\mu_2+\mu_3)\, P_{0,c_2,c_3,0,0} = \mu_1\alpha_{12} P_{1,c_2-1,c_3,0,0} + \mu_1\alpha_{13} P_{1,c_2,c_3-1,0,0} + +\mu_5 P_{0,c_2,c_3,0,1}$ (22)

For $c_1>b_1$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_5>0$

$(\lambda_1+\mu_1+\mu_5) P_{c_1,0,0,0,c_5} = \lambda_1 P_{c_1-b_1,0,0,0,c_5} + \mu_2 P_{c_1,1,1,0,c_5-1} + \mu_3 P_{c_1,0,1,0,c_5-1} + \mu_4 P_{c_1,0,0,1,c_5-1} + \mu_5 P_{c_1,0,0,0,c_5+1}$ (23)

For $c_1=0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_5>0$

$(\lambda_1+\mu_5)\, P_{0,0,0,0,c_5} = \mu_2 P_{0,1,1,0,c_5-1} + \mu_3 P_{0,0,1,0,c_5-1} + \mu_4 P_{0,0,0,1,c_5-1} + \mu_5 P_{0,0,0,0,c_5+1}$ (24)

For $c_1>0$, $c_2 = 0$, $c_3 = 0$, $c_4 > 0$, $c_5=0$

$(\lambda_1+\mu_1+\mu_4)\, P_{c_1,0,0,c_4,0} = \lambda_1 P_{c_1,0,0,c_4,0} + \mu_1\alpha_{14} P_{c_1+1,0,0,c_4-1,0} + \mu_5 P_{c_1,0,0,c_4,1}$ (25)

For $c_1=0$, $c_2 = 0$, $c_3 = 0$, $c_4 > 0$, $c_5=0$

$(\lambda_1+\mu_4)\, P_{0,0,0,c_4,0} = \mu_1\alpha_{14} P_{1,0,0,c_4-1,0} + \mu_5 P_{0,0,0,c_4,1}$ (26)

For $c_1>0$, $c_2 > 0$, $c_3 = 0$, $c_4 = 0$, $c_5=0$

$(\lambda_1+\mu_1+\mu_2)\, P_{c_1,c_2,0,0,0} = \lambda_1 P_{c_1,c_2,0,0,0} + \mu_1\alpha_{12} P_{c_1+1,c_2-1,0,0,0} + \mu_5 P_{0,c_2,0,0,1}$ (27)

For $c_1=0$, $c_2 > 0$, $c_3 = 0$, $c_4 = 0$, $c_5=0$

$(\lambda_1+\mu_2)\, P_{0,c_2,0,0,0} = \mu_1\alpha_{12} P_{c_1+1,c_2-1,0,0,0} + \mu_5 P_{0,c_2,0,0,1}$ (28)

For $c_1>0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_5=0$

$(\lambda_1+\mu_1)\, P_{c_1,0,0,0,0} = \lambda_1 P_{c_1,0,0,0,0} + \mu_5 P_{0,c_2,0,0,1}$ (29)

For $c_1=0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_5=0$

$(\lambda_1+\mu_1)\, P_{0,0,0,0,0} = \mu_5 P_{0,c_2,0,}$ (30)

For $c_1>0$, $c_2 = 0$, $c_3 > 0$, $c_4 = 0$, $c_5=0$

$(\lambda_1+\mu_1+\mu_3)\, P_{c_1,0,c_3,0,0} = \lambda_1 P_{c_1,0,c_3,0,0} + \mu_1\alpha_{13} P_{c_1+1,0,c_3-1,0,0} + \mu_5 P_{c_1,0,c_3,0,1}$ (31)

For $c_1=0$, $c_2 = 0$, $c_3 > 0$, $c_4 = 0$, $c_5=0$

$(\lambda_1+\mu_3) P_{0,0,c_3,0,0} = \mu_1\alpha_{13} P_{1,0,c_3-1,0,0} + \mu_5 P_{0,0,c_3,0,1}$ (32)

In order to solve the system of steady state equations (1) to (32) we apply here generating function technique. The generating function is defined as:

$F(X,Y,Z,R,S) = \sum_{c_1=0}^{\infty}\sum_{c_2=0}^{\infty}\sum_{c_3=0}^{\infty}\sum_{c_4=0}^{\infty}\sum_{c_5=0}^{\infty} P_{c_1,c_2,c_3,c_4,c_5} X^{c_1}Y^{c_2}Z^{c_3}R^{c_4}S^{c_5}$

Also for simplification we define partial generating function as:

$$\left.\begin{aligned}
F_{c_2,c_3,c_4,c_5}(X) &= \sum_{c_1=0}^{\infty} P_{c_1,c_2,c_3,c_4,c_5} X^{c_1}\\
F_{c_3,c_4,c_5}(X,Y) &= \sum_{c_2=0}^{\infty} F_{c_2,c_3,c_4,c_5}(X)\, Y^{c_2}\\
F_{c_4,c_5}(X,Y,Z) &= \sum_{c_3=0}^{\infty} F_{c_3,c_4,c_5}(X,Y)\, Z^{c_3}\\
F_{c_5}(X,Y,Z,R) &= \sum_{c_4=0}^{\infty} F_{c_4,c_5}(X,Y,Z)\, R^{c_4}\\
F(X,Y,Z,R,S) &= \sum_{c_5=0}^{\infty} F_{c_5}(X,Y,Z,R)\, S^{c_5}
\end{aligned}\right\} \quad (A')$$

After simplifying equations(1-32) by using generating function technique and calculus laws ,we get

$$F(X,Y,Z,R,S) = \frac{P}{Q} \quad \text{(I)}$$

Where

$$P = \mu_1\left(1-\frac{Y\alpha_{12}}{X}-\frac{\alpha_{13}Z}{X}-\frac{\alpha_{14}R}{X}\right)F_0(Y,Z,R,S)+\mu_2\left(1-\frac{S}{Y}\right)F_0(X,Z,R,S)+\mu_3\left(1-\frac{S}{Z}\right)F_0(X,Y,R,S)+\mu_4\left(1-\frac{S}{R}\right)F_0(X,Y,Z,S)+\mu_5\left(1-\frac{1}{S}\right)F_0(X,Y,Z,R)$$

$$Q = \left[\lambda_1(1-X)+\mu_1\left(1-\frac{Y\alpha_{12}}{X}-\frac{\alpha_{13}Z}{X}-\frac{\alpha_{14}R}{X}\right)+\mu_2\left(1-\frac{S}{Y}\right)+\mu_3\left(1-\frac{S}{Z}\right)+\mu_4\left(1-\frac{S}{R}\right)+\mu_5\left(1-\frac{1}{S}\right)\right]F(X,Y,Z,R,S)$$

For X=Y=Z=R=S=1 and F (1,1,1,1,1) = 1

For convenience we define: $F_0(Y, Z, R, S) = F_1;$
$F_0(X, Z, R, S) = F_2$
$F_0(X, Y, R, S) = F_3$
$F_0(X, Y, Z, S) = F_4;$
$F_0(X, Y, Z, R) = F_5;$

The equation (I) reduces to indeterminant form $\frac{0}{0}$,therefor by using L Hospital rule for limits the following results are obtained from equation (I) and by using the value of $F_1, F_2, F_3, F_4, F_5.$

a) when Y=Z=R=S=1 and taking X approaches to 1,we get $\mu_1F_1 = -\lambda_1+\mu_1$
b) when X=Z=R=S=1 and taking Y approaches to 1,we get $\mu_2F_2 = -\lambda_1\alpha_{34}+\mu_2$
c) When Y=X=R=S=1 and taking Z approaches to 1,we get $\mu_3F_3 = -\lambda_1+\mu_3$
d) when Y=X=Z=S=1 and taking R approaches to 1,we get $\mu_4F_4 = -\lambda_1\alpha_{14}+\mu_4$
e) when Y=X=Z=R=1 and taking S approaches to 1,we get $\mu_5F_5 = -\lambda_1\alpha_{12}-\lambda_1\alpha_{13}-\lambda_1\alpha_{14}+\mu_5$

on solving these equations for $F_1, F_2, F_3, F_4, F_5,$ we get

$$F_1=1-\frac{\lambda_1}{\mu_1} = 1-\rho_1$$
$$F_2=1-\frac{\lambda_1}{\mu_2}\alpha_{12} = 1-\rho_2$$
$$F_3= 1-\frac{\lambda_1}{\mu_3}\alpha_{13} = 1-\rho_3$$
$$F_4= 1-\frac{\lambda_1}{\mu_4}\alpha_{14} = 1-\rho_4$$
$$F_5= 1-\frac{\lambda_1}{\mu_5} = 1-\rho_5$$

Therefore,

$$P_{c_{1},\, c_{2},\, c_{3},\, c_{4},\, c_{5}}= {\rho_1}^{n_1}{\rho_2}^{n_2}{\rho_3}^{n_3}{\rho_4}^{n_4}{\rho_5}^{n_5}(1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4)(1-\rho_5)$$

Where $\rho_1 = \frac{\lambda_1}{\mu_1}$

$$\rho_2 = \frac{\lambda_1}{\mu_2}\alpha_{12}$$
$$\rho_3 = \frac{\lambda_1}{\mu_3}\alpha_{13}$$
$$\rho_2 = \frac{\lambda_1}{\mu_4}\alpha_{14}$$
$$\rho_5 = \frac{\lambda_1}{\mu_5}$$

The result is stable if $\rho_1\ \rho_2\ \rho_3\ \rho_4\ \rho_5 < 1$ is satisfied.

5.ALGORITHM:

The algorithm gives the plan of action to find the different queue characteristics like average queue length, variance of queue and average waiting time of the jobs for the development of a model consisting of three parallel service channels connected in series with the two services channels.

Step 1: Obtain the value of arrival rate λ_1,

Step 2: Obtain the value of service rate μ_1, μ_2, μ_3, μ_4 and μ_5

Step 3: Obtain the value of joining the previous server $\alpha_{12}, \alpha_{13}, \alpha_{14}$

Step 4: Find the mean queue length by using the formulae:

$$L= L_{q_1}+ L_{q_2} + L_{q_3} + L_{q_4} + L_{q_5}$$
$$L_{q_i} = \frac{\rho_i}{1-\rho_i} \quad (i = 1\ to\ 5)$$

Where $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ are service rate, λ_1, is arrival rate and α_{12}, α_{13} and α_{14} are the probabilities.

Step 5: Find the Variance of queues by using the formulae:

$$V=\frac{\rho_1}{(1-\rho_1)^2}+\frac{\rho_2}{(1-\rho_2)^2}+\frac{\rho_3}{(1-\rho_3)^2}+\frac{\rho_4}{(1-\rho_4)^2}+\frac{\rho_5}{(1-\rho_5)^2}$$
$$\rho_1 = \frac{\lambda_1}{\mu_1}$$
$$\rho_2 = \frac{\lambda_1}{\mu_2}\alpha_{12}$$
$$\rho_3 = \frac{\lambda_1}{\mu_3}\alpha_{13}$$
$$\rho_4 = \frac{\lambda_1}{\mu_4}\alpha_{14}$$
$$\rho_5 = \frac{\lambda_1}{\mu_5} \quad where,\ \alpha_{12} + \alpha_{13} + \alpha_{14} = 1$$

Step 6: Find the average waiting time of the customers using the formulae

$$E(W)= \frac{L}{\lambda_1}$$

6. QUEUE CHARACTERISTICS:

Mean queue length $L= L_{q_1}+ L_{q_2} + L_{q_3} + L_{q_4} + L_{q_5}$

Where ,

$$L_{q_1} = \frac{\rho_1}{1-\rho_1} = \frac{\lambda_1}{\mu_1 - \lambda_1}$$

$$L_{q_2} = \frac{\rho_2}{1-\rho_2} = \frac{\lambda_1\alpha_{12}}{\mu_2 - \lambda_1\alpha_{12}}$$
$$L_{q_3} = \frac{\rho_3}{1-\rho_3} = \frac{\lambda_1\alpha_{13}}{\mu_3 - \lambda_1\alpha_{13}}$$
$$L_{q_4} = \frac{\rho_4}{1-\rho_4} = \frac{\lambda_1\alpha_{14}}{\mu_4 - \lambda_1\alpha_{14}}$$
$$L_{q_5} = \frac{\rho_5}{1-\rho_5} = \frac{\lambda_1}{\mu_5 - \lambda_1}$$

The average no. of customers (mean queue length) is given by

$$L = L_{q_1} + L_{q_2} + L_{q_3} + L_{q_4} + L_{q_5}$$
$$= \frac{\lambda_1}{\mu_1-\lambda_1} + \frac{\lambda_1\alpha_{12}}{\mu_2-\lambda_1\alpha_{12}} + \frac{\lambda_1\alpha_{13}}{\mu_3-\lambda_1\alpha_{13}} + \frac{\lambda_1\alpha_{14}}{\mu_4-\lambda_1\alpha_{14}} + \frac{\lambda_1}{\mu_5-\lambda_1}$$

Now, variance of queue

$$V=\frac{\rho_1}{(1-\rho_1)^2} + \frac{\rho_2}{(1-\rho_2)^2} + \frac{\rho_3}{(1-\rho_3)^2} + \frac{\rho_4}{(1-\rho_4)^2} + \frac{\rho_5}{(1-\rho_5)^2}$$
$$= \frac{\lambda_1\mu_1}{(\mu_1-\lambda_1)^2} + \frac{\lambda_1\alpha_{12}\mu_2}{(\mu_2-\lambda_1)^2} + \frac{\lambda_1\mu_3\alpha_{13}}{(\mu_3-\lambda_1\alpha_{13})^2} + \frac{\lambda_1\mu_4\alpha_{14}}{(\mu_4-\lambda_1\alpha_{14})^2} + \frac{\lambda_1\mu_5}{(\mu_5-\lambda_1)^2}$$

7.**NUMERICAL ILLUSTRATION**:

The numerical is carried out to test the efficiency of the algorithm:

Sr.no.	Average service rate	Average arrival rate	Possibilities
1	$\mu_1 = 18$	$\lambda_1=4$	$\alpha_{12} = 0.3$
2	$\mu_2=11$		$\alpha_{13} = 0.3$
3	$\mu_3=20$		$\alpha_{14} = 0.4$
4	$\mu_4=9$		
5	$\mu_5=9$		

Find Average queue length, variance and average waiting time for jobs.

Here,

$$\rho_1 = \frac{\lambda_1}{\mu_1} = 0.2$$
$$\rho_2 = \frac{\lambda_1}{\mu_2}\alpha_{12} = 0.1$$
$$\rho_3 = \frac{\lambda_1}{\mu_3}\alpha_{13} = 0.06$$
$$\rho_4 = \frac{\lambda_1}{\mu_4}\alpha_{14} = 0.17$$
$$\rho_5 = \frac{\lambda_1}{\mu_5} = 0.44 \quad \alpha_{12} + \alpha_{13} + \alpha_{14} = 1$$

therefore on solving , we get mean queue length as :

$$L = L_{q_1} + L_{q_2} + L_{q_3} + L_{q_4} + L_{q_5}$$
$$= \frac{\lambda_1}{\mu_1-\lambda_1} + \frac{\lambda_1\alpha_{12}}{\mu_2-\lambda_1\alpha_{12}} + \frac{\lambda_1\alpha_{13}}{\mu_3-\lambda_1\alpha_{13}} + \frac{\lambda_1\alpha_{14}}{\mu_4-\lambda_1\alpha_{14}} + \frac{\lambda_1}{\mu_5-\lambda_1}$$
$$=0.28 + 0.122 + 0.063 + 0.21 + 0.8$$
$$=1.475$$

Variance of queues

$$V=\frac{\rho_1}{(1-\rho_1)^2} + \frac{\rho_2}{(1-\rho_2)^2} + \frac{\rho_3}{(1-\rho_3)^2} + \frac{\rho_4}{(1-\rho_4)^2} + \frac{\rho_5}{(1-\rho_5)^2}$$
$$=0.3125+0.123+0.0679+0.246+1.403$$
$$=2.1524$$

Average waiting time for customers

$$E(W)= \frac{L}{\lambda_1}=0.36875$$

8.CONCLUSION:

In the present paper we analyze a queue network model comprising of a parallel system commonly connected with two different service channels in series in stochastic environment. The model has been explained with the help of numerical illustration and providing the practical situation of a multi specialist hospital. Various queue characteristics like mean queue length, variance and average waiting time has been calculated by using Generating function technique.

Future Scope: The model can be extended by using more subservice channels.

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