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On Weak Forms Of B-Open And B-Closed Functions In Smooth Fuzzy Topological Spaces

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Abstract.

In this paper, we introduce and study two new classes of fuzzy functions by using the notions of r-fuzzy β-open sets and r-fuzzy β-closure operator called weakly r-fuzzy β-open and weakly r-fuzzy β-closed functions. The connections between these r-fuzzy functions and other existing r-fuzzy topological functions are studied.
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1. Introduction and Preliminaries.

Sˆostak [18] introduced the fuzzy topology as an extension of Chang's fuzzy topology [4] and developed in many directions [7, 8, 17]. In fuzzy topological spaces, a weaker forms of fuzzy continuity by many authors [1, 2, 3, 6, 20]. Kim and Park [9] introduced r-δ-cluster points and δ-closure operators in Sˆostak fuzzy topological spaces. Park et al. [11] introduced the concept of fuzzy semi-preopen sets which is weaker than any of fuzzy semi-open or fuzzy preopen sets. In 1968, Velicko [19] studied some new types of open sets called θ-open sets and δ-open sets. In 1963, Levine initiated a new type of open set called semi-open set [10]. In 1993, Raychaudhuri and Mukherjee defined δ-preopen sets [16] and δ-semiopen sets by Park [14]. In 2008, Caldas [5] obtained θ-semi-open sets. In 2006, Shafei introduced fuzzy θ-closed [21] and fuzzy θ-open sets. Recently, M.Palanisamy [12] & [13] introduced r-fuzzy Z * -Open Sets and r-fuzzy Z* -Continuity and r-fuzzy Z * -Open Sets and r-fuzzy Z* -Continuity sets in fuzzy topological spaces in the sense of Sˆostak's. In this paper, we introduce the concept of r-fuzzy β-open sets and r-fuzzy β-closure operator called weakly r-fuzzy β-open and weakly r-fuzzy β-closed functions. The connections between these r-fuzzy functions and other existing r-fuzzy topological functions are studied. Also, discuss about some characterizations and properties of these notions.

Throughout this article, we denote nonempty sets by X, Y etc., $I = [0,1]$ and $I_0 = (0,1]$.. For $\alpha \in 1$, $\overline{\alpha}(x) = \alpha$, $\forall x \in X$ A fuzzy point x_t for $t \in I_0$ is an element of I^X such that

$$x_t(y) = \begin{cases} t & if \quad y \text{ is equal to } x \\ 0 & if \quad y \text{ is not equal to } x. \end{cases}$$

Let $P_t(X)$ denote the set of all fuzzy points in X. A fuzzy point $x_t \in \mu$ iff $t < \mu(x)$. $\mu \in I^X$ is quasi-coincident with ν, denoted by $\mu q \upsilon$, if $\exists x \in X$ such that $\mu(x) + \upsilon(x) > 1$.

If μ is not quasi-coincident with υ, we denoted $\mu \overline{q} \upsilon$. If A is a subset of X, we define the characteristic function χ_A on X by

$\chi_t(x) = \begin{cases} 1 & if \quad x \in A \\ 0 \quad if & if \quad x \notin A. \end{cases}$ All notations and definitions will be standard in the fuzzy set theory.

Definition 2.2. [17]

A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:

(1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$,

(2) $\tau(\vee_{i\in\Gamma} v_i) \geq \wedge_{i\in\Gamma} \tau(v_i)$, for any $\{v_i\}_{i\in\Gamma} \subset I^X$,

(3) $\tau(v_1 \wedge v_2) \geq \tau(v_1 \wedge v_2)$, for any $\upsilon_1, \upsilon_2 \in I^X$.

The pair (X,τ) is called a fuzzy topological space or Sˆostak's fuzzy topological space or smooth topological space (for short, fts, sfts, sts).

Remark 2.3. [17]

Let (X,τ) be a fts. Then, for every $r \in I_0$, $\tau = \{\upsilon \in I^X : \tau(\upsilon) \geq r\}$ is a Change's fuzzy topology on X.

Theorem 2.4. [17]

Let (X,τ) be a sfts. Then for each $\mu \in I^X$, $r \in I_0$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows:

$C_\tau(\mu,r) = \wedge\{\upsilon \in I^X : \mu \leq \upsilon, \tau(\underline{1} - \upsilon) \geq r\}..$

For $\mu,\upsilon \in I^X$ and $r,s \in I_0$, the operator C_τ satisfies the following conditions:

(1) $C_\tau(\underline{0},r) = \underline{0},$

(2) $\lambda \leq C_\tau(\lambda,r),$

(3) $C_\tau(\lambda,r) \vee C_\tau(\mu,r) = C_\tau(\lambda \vee \mu,r),$

(4) $C_\tau(\lambda,r) \leq C_\tau(\mu,s)$ if $r \leq s,$

(5) $C_\tau(C_\tau(\lambda,r),r) = C_\tau(\mu,r).$

Theorem 2.5. [21]

Let (X,τ) be a sfts. Then for each $\mu \in I^X$, $r \in I_0$, we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows:

$I_\tau(\mu,r) = \vee\{\upsilon \in I^X : \mu \leq \upsilon, \tau(\upsilon) \geq r\}..$

For $\mu,\upsilon \in I^X$ and $r,s \in I_0$, the operator C_τ satisfies the following conditions:

(1) $I_\tau(\underline{1},r) = \underline{1},$

(2) $\lambda \geq I_\tau(\lambda,r),$

(3) $I_\tau(\lambda,r) \wedge I_\tau(\mu,r) = I_\tau(\lambda \wedge \mu,r),$

(4) $I_\tau(\lambda,r) \leq I_\tau(\mu,s)$ if $r \geq s,$

(5) $I_\tau(I_\tau(\lambda,r),r) = I_\tau(\lambda,r),$

(6) $I_\tau(\underline{1} - \lambda,r) = \underline{1} - C_\tau(\lambda,r)$ and $C_\tau(\underline{1} - \lambda,r) = \underline{1} - I_\tau(\lambda,r)$.

Definition 2.6. [13]

A point x of X is called δ -cluster point of λ if $I_\tau(C_\tau(U,r),r) \wedge \lambda = \phi,$ for everyopen set U of X containing x. The set of all δ -cluster point of λ is called δ -closure of λ and is denoted $\delta C_\tau(\lambda).$

Definition 2.6.
A set λ is δ -closed if and only if $\lambda = \delta C_\tau(\lambda, r).$ The complement of a δ -closed set is said to be δ -open [13]. Then δ -interior of a subset λ of X is the union of all δ -open sets of X contained in λ .

Definition 2.8.
Let (X,τ) be a sfts. Then for each $\lambda \in I^X$, $r \in I_0$, λ is called

1. r-fuzzy preopen (resp. r-fuzzy preclosed) [15] set if $\lambda \le I_\tau(C_\tau(\lambda,r),r),$
$(C_\tau(I_\tau(\lambda,r),r) \le \lambda),$
2. r-fuzzy α -open (resp. r-fuzzy α -closed) [15] set if $\lambda \le I_\tau(C_\tau(I_\tau(\lambda,r),r),r),$
$(C_\tau(I_\tau(C_\tau(\lambda,r),r),r) \le \lambda),$
3. r-fuzzy β -open (resp. r-fuzzy β -closed) [15] set if $\lambda \le C_\tau(I_\tau(C_\tau(\lambda,r),r),r)$
$(I_\tau(C_\tau(C_\tau(\lambda,r),r),r) \le \lambda),$
4. r-fuzzy θ -open (resp. r-fuzzy θ -closed) [5] set if $\lambda = \theta I_\tau(\lambda,r),$
$(\lambda \le \theta C_\tau(\lambda,r)),$

The family of all r-fuzzy preopen (resp. r -fuzzy α -open, r -fuzzy β -open, r -fuzzy θ -open) is denoted by $PO(X)$ (resp. $\alpha O(X)$, $eO(X)$, $\beta O(X)$, $\theta O(X)$.

Lemma: 2.1 [15]
Let λ, μ be two subsets of (X, τ) Then:
(1) λ is r-fuzzy δ -open if and only if $\lambda = I_\tau(\lambda,r)$,
(2) X - $\delta I_\tau(\lambda,r)$ = $\delta C_\tau(X \setminus (\lambda,r))$ and $\delta I_\tau(X \setminus (\lambda,r)) =$ X - $\delta I_\tau(\lambda,r)$,
(3) $I_\tau(\lambda,r) \le \delta C_\tau(\lambda,r)$ (resp. $\delta I_\tau(\lambda,r) \le I_\tau(\lambda,r)$)), for any subset λ of X,
(4) $\delta\text{-}C_\tau(\lambda \vee \mu, r) = \delta\text{-}C_\tau(\lambda,r) \vee \delta\text{-}C_\tau(\mu,r),\quad \delta\text{-}I_\tau(\lambda \vee \mu, r) = \delta\text{-}I_\tau(\lambda,r) \vee \delta\text{-}I_\tau(\mu,r),$

Definition
A space X is called r -fuzzy extremally disconnected (E.D) [19] if the closure of each r -fuzzy open set in X is open. A space X is called r -fuzzy β -connected [1, 14] if X cannot be expressed as the union of two nonempty disjoint r -fuzzy β -open sets.

Definition
A function $f:(X,\tau) \to (Y,\sigma)$ is called:
(i) r -fuzzy β -continuous [13] if for each r -fuzzy open subset μ of Y , $f^{-1}(\mu) \in \beta O(X,\tau)$..
(ii) r -fuzzy strongly continuous [3, 8], if for every fuzzy subset λ of X , $f(C_\tau(\lambda)) \le f(\lambda)$..
(iii) r -fuzzy weakly open [15] if $f(\eta) \le I_\tau(f(C_\tau(\eta,r),r))$ for each r -fuzzy open subset η of X .
(iv) r -fuzzy weakly closed [15] if $C_\tau(f(I_\tau(F,r),r)) \le f(F)$ for each r -fuzzy closed subset F of X .
(v) r -fuzzy relatively weakly open [15] provide that $f(\eta)$ is r -fuzzy open in $f(C_\tau(\eta,r))$ for every r -fuzzy open subset η of X .

(vi) r-fuzzy almost open, written as (a.o.S) [13] if the image of each r-fuzzy regular open subset η of X is r-fuzzy open set in Y.
(vii) r-fuzzy preopen [13] (resp. r-fuzzy β-open, r-fuzzy α-open) if for each r-fuzzy open subset η of X, $f(\eta)$ is r-fuzzy preopen (resp. $f(\eta)$ is r-fuzzy β-open, f(U) is r-fuzzy α-open) set in Y.
(viii) r-fuzzy preclosed [18] (resp. r-fuzzy β-closed , r-fuzzy α-closed) if for each r-fuzzy closed subset F of X, $f(F)$ is r-fuzzy preclosed (resp. $f(F)$ is r-fuzzy β-closed, $f(F)$ is r-fuzzy α-closed) set in Y.
(ix) r-fuzzy contra-open [13] (resp. r-fuzzy contra-closed [1], r-fuzzy contra β-closed) if $f(\eta)$ is r-fuzzy closed (resp. r-fuzzy open, r-fuzzy β-open) in Y for each r-fuzzy open (resp. r-fuzzy closed, r-fuzzy closed) r-fuzzy subset η of X.

2. Weakly r-fuzzy β-open functions.

In this section, we define the concept of r-fuzzy weak β-openness as a natural dual to the r-fuzzy weak β-continuity due to and Noiri [14] and we obtain several fundamental properties of this new function.

Definition 2.1.
A function $f:(X,\tau)\to(Y,\sigma)$ is said to be r-fuzzy weakly β-open if
$f(\eta)\le\beta(I_\tau(f(C_\tau(\eta,r),r)))$ for each r-fuzzy open set η of X.
Clearly, every r-fuzzy weakly open function is r-fuzzy weakly β-open and every r-fuzzy β-open function is also r-fuzzy weakly β-open, but the converse is not generally true. For,

Example 2.2.
Let $X=Y=\{a,b,c\}$ and $\mu_1,\mu_2\in I^X$ $\mu_2,\mu_3\in I^Y$, $\mu_5\in I^Z$ defined as follows:

$$\mu_1(a)=0.4,\mu_1(b)=0.5,\mu_1(c)=0.6,$$
$$\mu_2(a)=0.4,\mu_2(b)=0.5,\mu_2(c)=0.4,$$
$$\mu_3(a)=0.6,\mu_3(b)=0.5,\mu_3(c)=0.4,$$
$$\mu_4(a)=0.4,\mu_4(b)=0.5,\mu_4(c)=0.6,\ \mu_5(a)=0.6,\mu_5(b)=0.5,\mu_5(c)=0.4,$$

Define fuzzy topology $\tau,\sigma,\eta=I^X\to I$ as follows

$$\tau(\lambda)=\begin{cases}1 if\lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2} if\lambda=\mu_1,\\ \frac{1}{2} if\lambda=\mu_2,\\ 0 if\cdot otherwise\end{cases}$$

$$\sigma(\lambda)=\begin{cases}1 if\lambda\in\{\underline{0},\underline{1}\},,\\ \frac{1}{2} if\lambda=\mu_3,\\ \frac{1}{2} if\lambda=\mu_4,\\ \frac{1}{2} if\lambda=\mu_3\vee\mu_4,\\ \frac{1}{2} if\lambda=\mu_3\wedge\mu_4,\\ 0 if\cdot otherwise\end{cases}$$

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$$\eta(\lambda)=\begin{cases}1 if \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2} if \lambda=\mu_5,\\ 0 if \cdot otherwise\end{cases}$$

(i) Then the identity mapping $id_X:(X,\tau)\to(Y,\sigma)$ is $\frac{1}{2}$- fuzzy weakly β-open set but not $\frac{1}{2}$-fuzzy weakly open.

(ii) Then the identity mapping $id_X:(X,\tau)\to(Z,\eta)$ is $\frac{1}{2}$ fuzzy weakly β-open but not $\frac{1}{2}$-fuzzy β-open

Theorem 2.3.
Let X be a r-fuzzy regular space. Then $f:(X,\tau)\to(Y,\sigma)$ is r-fuzzy weakly β-open if and only if f is r-fuzzy β-open.

Proof.
The sufficiency is clear.
For the necessity, let χ be a nonempty r-fuzzy open subset of X. For each x in χ, let η_X be an r-fuzzy open set such that $x\in\eta_X\le C_\tau(\eta_X,r)\le\chi$. Hence we obtain that $\chi=\vee\{\eta_X:x\in\chi\}=\vee\{C_\tau(\eta_X,r):x\in\chi\}$ and,
$f(\chi)=\vee\{f(\eta_X):x\in\chi\}\le\vee\{\beta I_\tau(f(C_\tau(\eta_X,r)):x\in\chi\}\le\beta_\tau(f(\vee\{C_\tau(\eta_X,r):x\in\chi\})$
$=\beta I_\tau(f(\chi,r))$.
Thus f is r-fuzzy β-open.

Theorem 2.4.
For a function $f:(X,\tau)\to(Y,\sigma)$, the following conditions are equivalent :
(i) f is r-fuzzy weakly β-open.
(ii) $f(\theta I_\tau(\lambda,r))\le\beta I_\tau(f(\lambda,r))$ for every r-fuzzy subset λ of X.
(iii) $\theta I_\tau(f^{-1}(\mu,r))\le f^{-1}(\beta I_\tau(\mu,r))$ for every r-fuzzy subset μ of Y.
(iv) $f^{-1}(\beta C_\tau(\mu,r))\le\theta C_\tau(f^{-1}(\mu,r))$ for every r-fuzzy subset μ of Y.
(v) For each $x\in X$ and each r-fuzzy open subset ϑ of X containing x, there exists a r-fuzzy β-open set η containing $f(x)$ such that $\eta\le f(C_\tau(\vartheta,r))$.
(vi) $f(I_\tau(\omega,r))\le\beta I_\tau(f(\omega,r))$ for each r-fuzzy closed subset ω of X.
(vii) $f(\vartheta)\le\beta I_\tau(f(C_\tau(\vartheta,r),r))$ for each r-fuzzy open subset ϑ of X.
(viii) $f(I_\tau(C_\tau(\vartheta,r),r))\le\beta I_\tau(f(C_\tau(\vartheta,r),r))$ for each r-fuzzy preopen subset ϑ of X.
(X) If $f(\vartheta)\le\beta I_\tau(f(C_\tau(\vartheta,r),r))$ For each r-fuzzy α-open subset ϑ of X.

Proof. (i) → (ii) : Let λ be any r-fuzzy subset of X and $x\in\theta I_\tau(\lambda,r)$. Then, there exists an r-fuzzy open set ϑ such that $x\in\vartheta\le C_\tau(\vartheta)\le\lambda$. Then, $f(x)\in f(\vartheta)\le f(C_\tau(\vartheta,r))\le f(\lambda)$. Since f is r-fuzzy weakly β-open,
$f(\vartheta)\le\beta I_\tau(f(C_\tau(\vartheta,r),r))\le\beta I_\tau(f(\lambda,r))$. It implies that $f(x)\le\beta I_\tau(f(\lambda,r))$.

This shows that $x\in f^{-1}(\beta I_\tau(f(\lambda,r)))$. Thus $\theta I_\tau(\lambda,r)\le f^{-1}(\beta I_\tau(f(\lambda,r)))$, and so, $f(\theta I_\tau(\lambda,r))\le\beta I_\tau(f(\lambda,r))$.

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(ii) → (i) : Let ϑ be an r-fuzzy open set in X. As $\vartheta \le \theta I_\tau(C_\tau(\vartheta,r))$ implies, $f(\vartheta) \le f(\theta I_\tau(C_\tau(\vartheta,r),r)) \le \beta I_\tau(f(C_\tau(\vartheta,r),r))$. Hence f is r-fuzzy weakly β-open.

(ii) → (iii) : Let μ be any r-fuzzy subset of Y.

Then by (ii), $f(\theta I_\tau(f^{-1}(C_\tau(\mu,r),r))) \le \beta I_\tau(\vartheta,r)$.

Therefore $\theta I_\tau(f^{-1}(\mu,r)) \le f^{-1}(\beta I_\tau(\mu,r))$.

(iii) → (ii) : This is obvious.

(iii) → (iv) : Let μ be any r-fuzzy subset of Y. Using (iii), we have $X - \theta C_\tau(f^{-1}(\mu,r)) = \theta I_\tau(X - f^{-1}(\mu,r)) = \theta I_\tau(f^{-1}(Y-\mu,r))$

$\le f^{-1}(\beta I_\tau(Y-\mu,r)) = f^{-1}(Y - \beta C_\tau(\mu,r)) = X - (f^{-1}(\beta C_\tau(\mu,r))$.

Therefore, we obtain $f^{-1}(\beta C_\tau(\mu,r)) \le \theta C_\tau(f^{-1}(\mu,r))$.

(iv) → (iii) : Similarly we obtain, $X - f^{-1}(\beta I_\tau(\mu,r)) \le X - \theta I_\tau(f^{-1}(\mu,r))$,

for every r-fuzzy subset μ of Y, i.e., $\theta I_\tau(f^{-1}(\mu,r)) \le f^{-1}(\beta I_\tau(\mu,r))$.

. (i) → (v) : Let $x \in X$ and ϑ be an r-fuzzy open set in X with $x \in X$. Since f is r-fuzzy weakly β-open. $f(x) \in f(\vartheta) \le \beta I_\tau(f(C_\tau(\vartheta,r),r)$. Let $\eta = \beta I_\tau(f(C_\tau(\vartheta,r),r)$. Hence $\eta \le f(C_\tau(\vartheta,r))$., with η containing $f(x)$.

(v) → (i) : Let ϑ be an r-fuzzy open set in X and let $y \in f(\vartheta)$. It following from (v) $\eta \le f(C_\tau(\vartheta,r))$ for some η is r-fuzzy β-open in Y containing y. Hence we have, $y \in \eta \le \beta I_\tau(f(C_\tau(\vartheta,r),r))$. This shows that $f(\vartheta) \le \beta I_\tau(f(C_\tau(\vartheta,r),r))$,

i.e., f is a r-fuzzy weakly β-open function.

(i) → (vi) → (vii) → (viii) → (ix) → (i) : This is obvious.

Theorem 2.5.

Let $f:(X,\tau) \to (Y,\sigma)$ be a bijective function. Then the following statements are equivalent.

(i) f is r-fuzzy weakly β-open.

(ii) $\beta C_\tau(f(\vartheta,r)) \le f(C_\tau(\vartheta,r))$ for each ϑ is r-fuzzy open of X.

(iii) $\beta C_\tau(f(I_\tau(\omega,r),r)) \le f(\omega)$ for each ω is r-fuzzy closed in X.

Proof. (i) → (iii) : Let ω be a r-fuzzy closed set in X. Then we have $f(X-\omega) = Y - f(\omega) \le \beta I_\tau(f(C_\tau(X-\omega,r),r))$ and so $Y - f(\omega) \le Y - \beta C_\tau(f(I_\tau(\omega,r),r))$. Hence $\beta C_\tau(f(I_\tau(\omega,r),r)) \le f(\omega)$.

(iii) → (ii) : Let ϑ be a r-fuzzy open set in X. Since $C_\tau(\vartheta,r)$ is a r-fuzzy closed set and $\vartheta \le I_\tau(C_\tau(\vartheta,r),r)$ by (iii) we have $\beta C_\tau(f(\vartheta,r)) \le \beta C_\tau(f(I_\tau(C_\tau(\vartheta,r),r),r)) \le f(C_\tau(\vartheta,r))$.

(ii) → (iii) : Similar to (iii) → (ii) .

(iii) → (i) : Clear.

Theorem 2.6.

If $f:(X,\tau) \to (Y,\sigma)$ is r-fuzzy weakly β-open and r-fuzzy strongly continuous, then f is r-fuzzy β-open.

Proof. Let ϑ be an r-fuzzy open subset of X. Since f is r-fuzzy weakly β-open $f(\vartheta) \le \beta I_\tau(f(C_\tau(\vartheta,r),r)))$. However, because f is r-fuzzy strongly continuous, $f(\vartheta) \le \beta I_\tau(f(\vartheta,r))$ and therefore $f(\vartheta)$ is r-fuzzy β-open.

Example 2.7.

A r -fuzzy β-open function need not be r -fuzzy strongly continuous.

Let X = {a, b, c}, and let τ be the indiscrete topology for X. Then the identity function of (X, τ) onto (X, τ) is a β-open function (hence weakly β-open function) which is not strongly continuous.

Theorem 2.8.

A function $f:(X,\tau)\to(Y,\sigma)$ is r -fuzzy β -open if f is r -fuzzy weakly β -open and relatively r -fuzzy weakly open.

Proof.

Assume f is r -fuzzy weakly β -open and relatively r -fuzzy weakly open. Let ϑ be an r -fuzzy open subset of X and let $y\in f(\vartheta)$. Since f is relatively r -fuzzy weakly open, there is an open subset η of Y for which $f(\vartheta)=f(C_\tau(\vartheta,r))\wedge\eta$. Because f is r -fuzzy weakly β -open, it follows that $f(\vartheta)\le\beta I_\tau(f(C_\tau(\vartheta,r),r))$. Then $y\in\beta I_\tau(f(C_\tau(\vartheta,r),r))\wedge\eta\le f(C_\tau(\vartheta,r))\wedge\eta=f(\vartheta)$ and therefore $f(\vartheta)$ is r -fuzzy β -open.

Theorem 2.9.

If $f:(X,\tau)\to(Y,\sigma)$ is r -fuzzy contra β -closed, then f is a r -fuzzy weakly β open function.

Proof.

Let ϑ be an r -fuzzy open subset of X. Then, we have $f(\vartheta)\le f(C_\tau(\vartheta,r))=\beta I_\tau(f(C_\tau(\vartheta,r),r))$. The converse of Theorem 2.9 does not hold.

Example 2.10.

A r -fuzzy weakly β -open function need not be r -fuzzy contra β -closed is given from Example 2.2(ii). Next, we define a dual form, called complementary r -fuzzy weakly β -open function.

Definition 2.11.

A function $f:(X,\tau)\to(Y,\sigma)$ is called complementary r -fuzzy weakly β --open (written as c.w.β.o) if for each r -fuzzy open set ϑ of X, $f(Fr(\vartheta,r))$ is r -fuzzy β -closed in Y, where $Fr(\vartheta,r)$ denotes the frontier of ϑ.

Example 2.12.

A r -fuzzy weakly β -open need not be r -fuzzy complementary weakly β -open.

Let $X=\{a,b,c\}, Y=\{a,b\}$ and $\mu_1,\mu_2,\mu_3\in I^X$ $\mu_4,\mu_5,\mu_6\in I^Y$ defined as follows:

$\mu_1(a)=0.4,\mu_1(b)=0.6,\mu_1(c)=0.6,$

$\mu_2(a)=0.4,\mu_2(b)=0.5,\mu_2(c)=0.6,$

$\mu_3(a)=0.6,\mu_3(b)=0.6,\mu_3(c)=0.6,$

$\mu_4(a)=0.6,\mu_4(b)=0.6,\mu_4(c)=0.5$

$\mu_5(a)=0.5,\mu_5(b)=0.5,\mu_4(c)=0.4$

$\mu_6(a)=0.5,\mu_6(b)=0.4.\mu_4(c)=0.4,$

Define fuzzy topology $\tau,\sigma=I^X\to I$ as follows

$$\tau(\lambda)=\begin{cases}1 if \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2} if \lambda=\mu_1,\\ \frac{1}{2} if \lambda=\mu_2,\\ \frac{1}{2} if \lambda=\mu_3,\\ 0 if \cdot otherwise\end{cases}\quad \sigma(\lambda)=\begin{cases}1 if \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2} if \lambda=\mu_4,\\ \frac{1}{2} if \lambda=\mu_5,\\ \frac{1}{2} if \lambda=\mu_6,\\ 0 if \cdot otherwise\end{cases}$$

Then the identity mapping $id_X:(X,\tau)\to(Y,\sigma)$ is $\frac{1}{2}$-fuzzy weakly β-open but not $\frac{1}{2}$-fuzzy complementary weakly β-open.

Example 2.13.

r-fuzzy complementary weakly β-open does not imply r-fuzzy weakly β-open.

Let $X=\{a,b\}$ and $\lambda_1,\lambda_2,\lambda_3,\lambda_4\in I^X$ $\mu_1,\mu_2\in I^Y$ defined as follows:

$\lambda_1(a)=0.3,\lambda_1(b)=0.4,\lambda_1(c)=0.5,$

$\lambda_2(a)=0.6,\lambda_2(b)=0.5,\lambda_2(c)=0.5,$

$\lambda_3(a)=0.6,\lambda_3(b)=0.5,\lambda_3(c)=0.4,$

$\lambda_4(a)=0.3,\lambda_4(b)=0.4,\lambda_4(c)=0.4,$

$\mu_1(a)=0.4,\mu_1(b)=0.7,\mu_1(c)=0.5,$

$\mu_2(a)=0.4,\mu_2(b)=0.1,\mu_2(c)=0.5.$

Define fuzzy topology $\tau=I^X\to I$ as follows

$$\tau(\lambda)=\begin{cases}1 if \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2} if \lambda=\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\},\\ 0 if \cdot otherwise.\end{cases}$$

$$\sigma(\lambda)=\begin{cases}1 if \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2} if \lambda=\mu_1,\\ \frac{1}{2} if \lambda=\mu_2,\\ 0 if \cdot otherwise\end{cases}$$

Then the identity mapping $id_X : (X,\tau) \to (Y,\sigma)$ is $\frac{1}{2}$-fuzzy weakly β-open but not $\frac{1}{2}$-fuzzy complementary weakly β-open.

Note:

Examples 2.12 and 2.13 demonstrate the independence of complementary $\frac{1}{2}$-fuzzy weakly β-openness and $\frac{1}{2}$-fuzzy weakly β-openness.

Theorem 2.14.

Let $\beta O(X,\tau)$ is r-fuzzy closed under intersections. If $f:(X,\tau)\to(Y,\sigma)$ is bijective r-fuzzy weakly β-open and c.w.β.o, then f is r-fuzzy β-open.

Proof.

Let ϑ be an r-fuzzy open subset in X with $x \in \vartheta$, since f is r-fuzzy weakly β-open, by Theorem 2.3(v) there exists a r-fuzzy β-open set η containing $f(x)=y$ such that $\eta \le f(C_\tau(\vartheta,r))$. Now $Fr(\vartheta,r)=C_\tau(\vartheta,r)-\vartheta$ and thus $x \notin Fr(\vartheta,r)$.. Hence $y \notin Fr(\vartheta,r)$. and therefore $y \in \eta - f(Fr(\vartheta,r))$. Put $\eta_y = \eta - f(Fr(\vartheta,r))$. a r-fuzzy β-open set since f is c.w.β.o. Since $y \in \eta_y, y \in f(Fr(\vartheta,r))$. But $y \notin f(Fr(\vartheta,r))$ and thus $y \notin f(Fr(\vartheta,r)) = f(C_\tau(\vartheta,r)) - f(\vartheta)$ which implies that $y \in f(\vartheta)$. Therefore $f(\vartheta)=\vee\{\eta_y : \eta_y \in \beta O(Y,\sigma), y \in f(\vartheta)\}$. Hence f is r-fuzzy β-open.

Theorem 2.15.

If $f:(X,\tau)\to(Y,\sigma)$ is an a.o.S function, then it is a r-fuzzy weakly β-open function.

Proof.

Let ϑ be an r-fuzzy open set in X. Since f is a.o.S and $I_\tau(C_\tau(\vartheta,r),r))$ is r-fuzzy regular open, $f(I_\tau(C_\tau(\vartheta,r),r))$ is r-fuzzy open in Y and hence $f(\vartheta) \le f(I_\tau(C_\tau(\vartheta,r),r)) \le I_\tau(f(C_\tau(\vartheta,r),r) \le \beta I_\tau(f(C_\tau(\vartheta,r),r))$. This shows that f is r-fuzzy weakly β-open. The converse of Theorem 2.15 is not true in general.

Example 2.16.

A r-fuzzy weakly β-open function need not be r-fuzzy a.o.S.

Let $X=\{a,b,c\}, Y=\{a,b,c\}$ and $\mu_1,\mu_2 \in I^X$ $\nu_1,\nu_2 \in I^Y$ defined as follows:

$\mu_1(a)=0.3, \mu_1(b)=0.2, \mu_1(c)=0.7,$

$\mu_2(a)=0.8, \mu_2(b)=0.8, \mu_2(c)=0.4,$

$\upsilon_1(a)=0.8, \upsilon_1(b)=0.7, \upsilon_1(c)=0.6,$

$\upsilon_2(a)=0.5, \upsilon_2(b)=0.6, \upsilon_2(c)=0.2.$

Define fuzzy topology $\tau = I^X \to I$ as follows

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$$\tau(\lambda)=\begin{cases}1 if \lambda\in\{\underline{0},\underline{1}\}, \\ \frac{1}{2} if \lambda=\mu_1, \\ \frac{1}{3} if \lambda=\mu_2, \\ \frac{2}{3} if \lambda=\mu_1\vee\mu_2, \\ \frac{2}{3} if \lambda=\mu_1\wedge\mu_2, \\ 0 if \cdot otherwise\end{cases}$$

$$\sigma(\lambda)=\begin{cases}1 if \lambda\in\{\underline{0},\underline{1}\}, \\ \frac{1}{2} if \lambda=\nu_1, \\ \frac{1}{2} if \lambda=\nu_2, \\ 0 if \cdot otherwise\end{cases}$$

Then the identity mapping $id_X:(X,\tau)\to(Y,\sigma)$ is $\frac{1}{2}$-fuzzy weakly β-open but not $\frac{1}{2}$-fuzzy a.o.S, since $I_\tau(f(I_\tau(C_\tau(\lambda,\frac{1}{2}),\frac{1}{2}),\frac{1}{2})=\phi$.

Lemma 2.17.
If $f:(X,\tau)\to(Y,\sigma)$ is a r-fuzzy continuous function, then for any r-fuzzy subset ϑ of X, $f(C_\tau(\vartheta,r))\le C_\tau(f(\vartheta,r))$.

Theorem 2.18. If $f:(X,\tau)\to(Y,\sigma)$ is a r-fuzzy weakly β-open and r-fuzzy continuous function, then f is a r-fuzzy β-open function.

Proof.
Let ϑ be a r-fuzzy open set in X. Then by r-fuzzy weak β-openness of f, $f(\vartheta)\le\beta I_\tau(f(C_\tau(\vartheta,r),r))$. Since f is r-fuzzy continuous $f(C_\tau(\vartheta,r))\le C_\tau(f(\vartheta,r))$. Hence we obtain that, $f(\vartheta)\le\beta I_\tau(f(C_\tau(\vartheta,r),r))\le\beta I_\tau(C_\tau(f(\vartheta,r),r))\le C_\tau(I_\tau(C_\tau(f(\vartheta,r),r),r))$. Therefore, $f(\vartheta)\le C_\tau(I_\tau(f(\vartheta,r)))$ which shows that $f(\vartheta)$ is a r-fuzzy β-open set in Y. Thus, f is a r-fuzzy β-open function.
Since every r-fuzzy strongly continuous function is r-fuzzy continuous we have the following corollary.

Corollary 2.19.
If $f:(X,\tau)\to(Y,\sigma)$ is an injective r-fuzzy weakly β-open and r-fuzzy strongly continuous function. Then f is r-fuzzy β-open.

Theorem 2.20.
If $f:(X,\tau)\to(Y,\sigma)$ is a injective r-fuzzy weakly β-open function of a space X onto a r-fuzzy β–connected space Y, then X is r-fuzzy connected.

Proof.
Let us assume that X is not r-fuzzy connected. Then there exist nonempty r-fuzzy open sets ϑ_1 and ϑ_2 such that $\vartheta_1\wedge\vartheta_2=\phi$ and $\vartheta_1\vee\vartheta_2=X$. Hence we have $f(\vartheta_1)\wedge f(\vartheta_2)=\phi$ and $f(\vartheta_1)\vee f(\vartheta_2)=Y$. Since f is r-fuzzy weakly β-open, we have $f(\vartheta_i)\leq\beta I_\tau(f(C_\tau(\vartheta_i,r),r)$ for $i=1,2$ and since ϑ_i is r-fuzzy open and also r-fuzzy closed, we have $f(C_\tau(\vartheta_i,r))=f(\vartheta_i)$ for $i=1,2$. Hence $f(\vartheta_i)$ is r-fuzzy β-open in Y for $i=1,2$. Thus, Y has been decomposed into two non-empty disjoint r-fuzzy β-open sets . This is contrary to the hypothesis that Y is a r-fuzzy β–connected space. Thus X is connected.
Recall, that a space X is said to be hyperconnected [12, 13] if every nonempty open subset of X is dense in X.

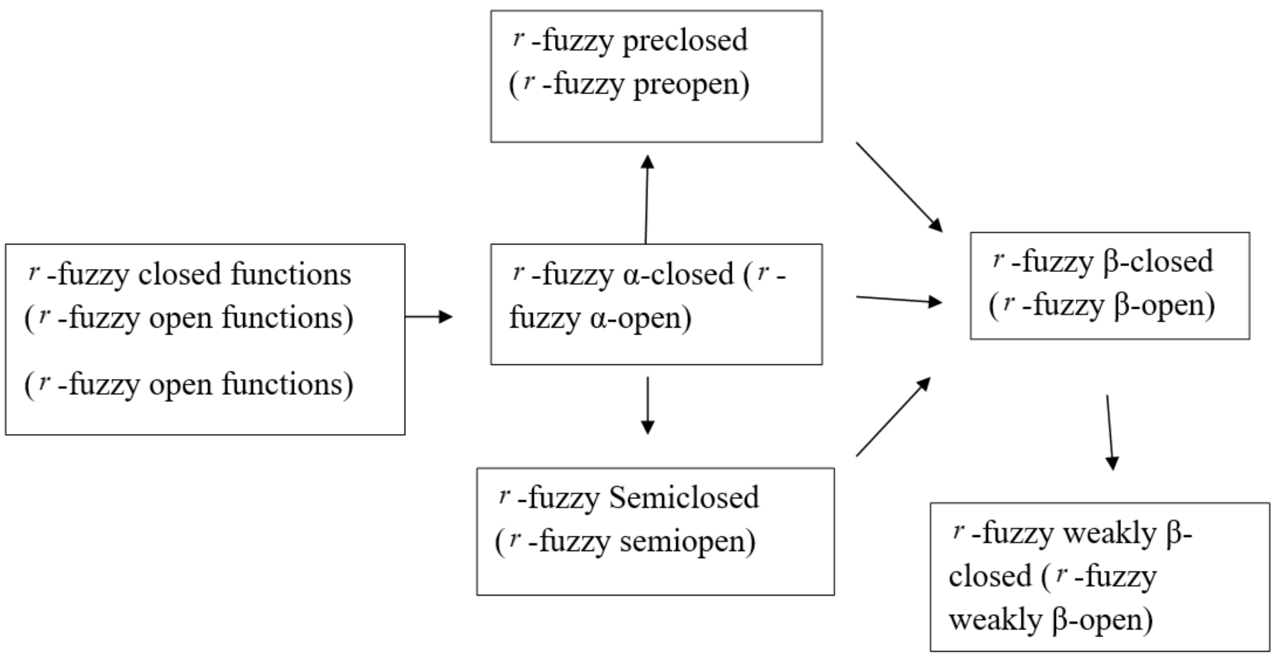
Theorem 2.21.
If X is a r-fuzzy hyperconnected space, then a function $f:(X,\tau)\to(Y,\sigma)$ is r-fuzzy weakly β-open if and only if $f(X)$ is r-fuzzy β-open in Y.

Proof.
The sufficiency is clear. For the necessity observe that for any r-fuzzy open subset ϑ of X, $f(\vartheta)\leq f(X)=\beta I_\tau(f(X,r))=\beta I_\tau(f(C_\tau(\vartheta,r),r))$.

3. r-fuzzy Weakly β-closed functions.

Now, we define the generalized form of r-fuzzy β-closed functions

Definition 3.1.
A function $f:(X,\tau)\to(Y,\sigma)$ is said to be r-fuzzy weakly β-closed if $\beta C_\tau(f(I_\tau(F,r),r))\leq f(F)$ for each closed set F in X.
The implications between r-fuzzy weakly β-closed (res. r-fuzzy weakly β-open) functions and other types of r-fuzzy closed (resp. r-fuzzy open) functions are given by the following diagram.



The converse of these statements are not necessarily true, as shown by the following examples.

Example.3.2
An injective function from a fuzzy discrete space into an fuzzy indiscrete space is r -fuzzy β-open and r -fuzzy β-closed, but neither r -fuzzy α-open nor r -fuzzy α-closed.

Example.3.3
Let $X=\{a,b\}$ and $Y=\{p,q\}$ .Define $\lambda_1,\lambda_2\in I^X$ $\mu_1,\mu_2\in I^Y$ as follows:

$$\lambda_1(a)=0.1,\lambda_1(b)=0.2,$$
$$\lambda_2(a)=0.2,\lambda_2(b)=0.1,$$
$$\mu_1(p)=0.6,\mu_1(q)=0.7,$$
$$\mu_2(p)=0.7,\mu_2(q)=0.6.$$

Define fuzzy topology $\tau_1,\tau_2=I^X\to I$ as follows

$$\tau_1(\lambda)=\begin{Bmatrix} 1 if \lambda\in\{\underline{0},\underline{1}\}, \\ \frac{2}{3} if \lambda=\lambda_1, \\ \frac{1}{4} if \lambda=\lambda_2, \\ \frac{1}{4} if \lambda=\lambda_1\vee\lambda_2, \\ 0 if\cdot otherwise \end{Bmatrix}$$

$$\tau_2(\mu)=\begin{Bmatrix} 1 if \mu\in\{\underline{0},\underline{1}\}, \\ \frac{2}{3} if \mu=\mu_1, \\ \frac{1}{4} if \mu=\mu_2, \\ \frac{1}{4} if \mu=\mu_1\vee\mu_2 \\ 0 if\cdot otherwise \end{Bmatrix}$$

Then the mapping $f:(X,\tau_1)\to(Y,\tau_2)$ define by $f(a)=q,f(b)=q.$ Then f is $\frac{2}{3}$ -fuzzy α -open and $\frac{2}{3}$ -fuzzy α -closed but neither $\frac{2}{3}$ -fuzzy open nor $\frac{2}{3}$ -fuzzy closed

Example 3.4.
Let f : (X, τ) → (Y, σ) be the function from Example 2.2. Then it is shown that f is r-fuzzy weakly β -closed which is not r-fuzzy weakly closed.
Example 3.5.

Let $X = \{a,b,c\}$ and $Y = \{p,q,r\}$.Define $\lambda_1, \lambda_2 \in I^X$ $\mu_1, \mu_2 \in I^Y$ as follows:

$\lambda_1(a) = 0.3, \lambda_1(b) = 0.7, \lambda_1(c) = 0.7$

$\lambda_2(a) = 0.7, \lambda_2(b) = 0.3, \lambda_2(c) = 0.3,$

$\mu_1(p) = 0.1, \mu_1(q) = 0.3, \mu_1(r) = 0.3$

$\mu_2(p) = 0.3, \mu_2(q) = 0.1, \mu_2(q) = 0.1.$

Define fuzzy topology $\tau_1, \tau_2 = I^X \to I$ as follows

$$\tau_1(\lambda) = \begin{cases} 1 if \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} if \lambda = \lambda_1, \\ \frac{1}{3} if \lambda = \lambda_2, \\ \frac{1}{3} if \lambda = \lambda_1 \vee \lambda_2, \\ 0 if \cdot otherwise \end{cases}$$

$$\tau_2(\mu) = \begin{cases} 1 if \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} if \mu = \mu_1, \\ \frac{1}{3} if \mu = \mu_2, \\ \frac{1}{3} if \mu = \mu_1 \vee \mu_2 \\ 0 if \cdot otherwise \end{cases}$$

Then the mapping $f : (X, \tau_1) \to (Y, \tau_2)$ define by $f(a) = q, f(b) = q, f(c) = r.$ Then f is $\frac{1}{2}$ -fuzzy β -closed ($\frac{1}{2}$ -fuzzy β -open) but not $\frac{1}{2}$ -fuzzy semi open ($\frac{1}{2}$ -fuzzy semi closed) .

Example 3.6.

Let $X = \{a,b,c\}$ and $Y = \{p,q,r\}$.Define $\lambda_1, \lambda_2 \in I^X$ $\mu_1 \in I^Y$ as follows:

$\lambda_1(a) = 0.4, \lambda_1(b) = 0.6, \lambda_1(c) = 0.4$

$\lambda_2(a) = 0.6, \lambda_2(b) = 0.4, \lambda_2(c) = 0.4,$

$\mu_1(p) = 0.6, \mu_1(q) = 0.4, \mu_1(r) = 0.5$

Define fuzzy topology $\tau_1, \tau_2 = I^X \to I$ as follows

$$\tau_1(\lambda) = \begin{cases} 1 if \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2} if \lambda = \lambda_1, \\ \frac{1}{2} if \lambda = \lambda_2, \\ \frac{1}{2} if \lambda = \lambda_1 \vee \lambda_2, \\ \frac{1}{2} if \lambda = \lambda_1 \wedge \lambda_2 \\ 0 if \cdot otherwise \end{cases}$$

$$\tau_2(\mu) = \begin{cases} 1 if \mu \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2} if \mu = \mu_1, \\ 0 if \cdot otherwise \end{cases}$$

Then the identity mapping $id_X : (X,\tau_1) \to (Y,\tau_2)$ define by $f(a) = a, f(b) = b, f(c) = c.$ Then f is $\frac{1}{2}$ -fuzzy β-closed ($\frac{1}{2}$ -fuzzy β-open) but not $\frac{1}{2}$ -fuzzy preclosed ($\frac{1}{2}$ -fuzzy preopen) .

Theorem 3.4.
For a function $f:(X,\tau) \to (Y,\sigma)$, the following conditions are equivalent.
(i) f is r-fuzzy weakly β-closed.
(ii) $\beta C_\tau(f(\vartheta)) \le f(C_\tau(\vartheta))$ for every r-fuzzy open set ϑ of X.
(iii) $\beta C_\tau(f(\vartheta)) \le f(C_\tau(\vartheta))$ for each r-fuzzy regular open subset ϑ of X,
(iv) For each subset F in Y and each r-fuzzy open set F in X with $f^{-1}(F) \le \vartheta$, ,there exists a r-fuzzy β-open set λ in Y with $F \le \lambda$ and $f^{-1}(F) \le C_\tau(\vartheta)$, ,
(v) For each point y in Y and each open set ϑ in X with $f^{-1}(y) \le \vartheta$, , there exists a r-fuzzy β-open set λ in Y containing y and $f^{-1}(\lambda) \le \vartheta$, ,
(vi) $\beta C_\tau(f(I_\tau(C_\tau(\vartheta,r),r),r) \le f(C_\tau(\vartheta,r))$ for each set ϑ in X,
(vii) $\beta C_\tau(f(I_\tau(\theta C_\tau(\vartheta,r),r),r) \le f(\theta C_\tau(\vartheta,r))$ for each set ϑ in X,
(viii) $\beta C_\tau(f(\vartheta,r),r) \le f(C_\tau(\vartheta,r))$ for each r-fuzzy β-open set ϑ in X.

Proof. $(i) \to (ii)$. Let ϑ be any r-fuzzy open subset of X. Then $\beta C_\tau(f(\vartheta,r)) = \beta C_\tau(f(I_\tau(\vartheta,r),r) \le \beta C_\tau(f(I_\tau(C_\tau(\vartheta,r),r),r) \le f(C_\tau(\vartheta,r))$.
$(ii) \to (i)$. Let F be any r-fuzzy closed subset of X. Then,
$\beta C_\tau(f(I_\tau(\vartheta,r),r) \le f(C_\tau(I_\tau(F,r),r) \le f(C_\tau(F,r)) = f(F)$.
It is clear that: $(ii) \to (vii)$, $(iv) \to (v)$, and $(i) \to (vi) \to (viii) \to (iii) \to (i)$.

To show that $(iii) \to (iv)$, : Let F be a fuzzy subset in Y and let ϑ be fuzzy open in X with $f^{-1}(F) \le \vartheta.$ Then $f^{-1}(F) \wedge C_\tau(X - C_\tau(\vartheta,r),r) = \phi.$ and consequently, $F \wedge f \ (C_\tau(X - C_\tau(\vartheta,r),r) = \phi.$. Since $X - C_\tau(\vartheta,r)$ is r-fuzzy regular open, F $F \wedge \beta C_\tau(f(X - C_\tau(\vartheta,r),r) = \phi$ by (iii).

Let $\lambda = Y - \beta C_{\tau}(f(X - C_{\tau}(\vartheta, r), r)..$ Then λ is r-fuzzy β-open with $F \leq \lambda$ and $f^{-1}(\lambda) \leq X - f^{-1}(\beta C_{\tau}(f(X - C(\vartheta, r), r)) \leq X - f^{-1} f(X - C_{\tau}(\vartheta, r)) \leq C_{\tau}(\vartheta, r).$

$(vii) \to (i)$: It is suffices see that $\theta C_{\tau}(\vartheta, r) = C_{\tau}(\vartheta, r)$ for every open sets ϑ in X.

$(v) \to (i)$: Let F be closed in X and let $y \in Y - f(F)$. Since $f^{-1}(y) \leq X - F$, there exists a r-fuzzy β-open λ in Y with $y \in \lambda$ and $f^{-1}(\lambda) \leq C_{\tau}(X - F, r) = X - I_{\tau}(F, r)$ by (v). Therefore $\lambda \wedge f(I_{\tau}(F, r) = \phi,$ so that $y \in Y - \beta C_{\tau}(f(I_{\tau}(F, r), r)..$ Thus $(v) \to (i)$. Finally, for $(vii) \to (viii)$: Note that $\theta C_{\tau}(\vartheta, r) = C_{\tau}(\vartheta, r)$ for each r-fuzzy β-open subset ϑ in X. The following theorem the proof is mostly straightforward and is omitted

Theorem 3.5.

For a function $f : (X, \tau) \to (Y, \sigma)$ the following conditions are equivalent :

(i) f is r-fuzzy weakly β-closed,

(ii) $\beta C_{\tau}(f(I_{\tau}(F, r), r) \leq f(F)$ for each r-fuzzy β-closed subset F in X,

(iii) $\beta C_{\tau}(f(I_{\tau}(F, r), r) \leq f(F)$ for every r-fuzzy α-closed subset F in X.

Remark 3.6.

By Theorem 2.5, if $f : (X, \tau) \to (Y, \sigma)$ is a bijective function, then f is r-fuzzy weakly β-open if and only if f is r-fuzzy weakly β-closed. Next we investigate conditions under which r-fuzzy weakly β-closed functions are r-fuzzy β-closed.

Theorem 3.7.

If $f : (X, \tau) \to (Y, \sigma)$ is r-fuzzy weakly β-closed and if for each fuzzy closed subset F of X and each fiber $f^{-1}(y) \leq X - F$ there exists a open ϑ of X such that $f^{-1}(y) \leq \vartheta \leq C_{\tau}(\vartheta, r) \leq X - F$. Then f is r-fuzzy β-closed.

Proof. Let F is any closed subset of X and let $y \in Y - f(F)$. Then $f^{-1}(y) \wedge F = \phi$ and hence $f^{-1}(y) \leq X - F..$ By hypothesis, there exists a open ϑ of X such that $f^{-1}(y) \leq \vartheta \leq C_{\tau}(\vartheta, r) = X - F..$ Since f is r-fuzzy β -weakly -closed by Theorem 3.4, there exists a r-fuzzy β-open η in Y with $y \in \eta$ and $f^{-1}(\eta) \leq C_{\tau}(\vartheta, r)$. Therefore, we obtain $f^{-1}(\eta) \wedge F = \phi$ and hence $\eta \wedge f(F) = \phi$, this shows that $y \notin \beta C_{\tau} f(F, r)$. Therefore, $f(F)$ is r-fuzzy β-closed in Y and f is r-fuzzy β-closed.

Theorem 3.8.

If $f : (X, \tau) \to (Y, \sigma)$ is contra-open, then f is weakly β-closed.

Proof. Let F be a closed subset of X. Then, $\beta C_{\tau} f(I_{\tau}(F, r), r) \leq f(I_{\tau}(F, r)) \leq f(F)$.

Theorem 3.9.

If $f : (X, \tau) \to (Y, \sigma)$ is one-one and r-fuzzy weakly β-closed, then for every subset F of Y and every open set ϑ in X with $f^{-1}(F) \leq \vartheta$, there exists a r-fuzzy β-closed set μ in Y such that $F \leq \mu$ and $f^{-1}(\mu) \leq C_{\tau}(\vartheta, r)$.

Proof.

Let F be a subset of Y and let ϑ be a open subset of X with $f^{-1}(F) \leq \vartheta$. Put $\mu = \beta C_{\tau}(f(I_{\tau}(C_{\tau}(\vartheta, r), r), r)$, then μ is a r-fuzzy β-closed subset of Y such that $F \leq \mu$ since

$F \leq f(\vartheta) \leq f(I_\tau(C_\tau(\vartheta,r),r) \leq \beta C_\tau(f(I_\tau(C_\tau(\vartheta,r),r),r) = \mu$. And since f is r-fuzzy weakly β-closed, $f^{-1}(\mu) \leq C_\tau(\vartheta,r)$.

Taking the set F in Theorem 3.9 to be y for $y \in Y$ we obtain the following result.

Corollary 3.10.

If $f:(X,\tau) \to (Y,\sigma)$ is one-one and r-fuzzy weakly β-closed, then for every point y in Y and every open set ϑ in X with $f^{-1}(y) \leq \vartheta$, there exists a r-fuzzy β-closed set μ in Y containing y such that $f^{-1}(\mu) \leq C_\tau(\vartheta,r)$. Recall that, a set F in a space X is r-fuzzy θ-compact if for each cover Ω of F by open ϑ in X, there is a finite family $\vartheta_1, \vartheta_2.\vartheta_3,\ldots,\vartheta_n$ in Ω such that $F \leq I_\tau(\vee\{C_\tau(\vartheta_i\} : i = 1,2,\ldots,n\})$ [16].

Theorem 3.11.

If $f:(X,\tau) \to (Y,\sigma)$ is r-fuzzy weakly β-closed with all fibers θ-closed, then $f(F)$ is r-fuzzy β-closed for each θ-compact F in X.

Proof.

Let F be r-fuzzy θ-compact and let $y \in Y - f(F)$. Then $f^{-1}(y) \wedge F = \phi$ and for each $x \in F$ there is an open U_x containing x in X and $C_\tau(\vartheta_x) \wedge f^{-1}(y) = \phi.$. Clearly $\Omega = \{U_x : x \in F\}$ is an open cover of F and since F is r-fuzzy θ-compact, there is a finite family $\{\vartheta_{x1}, \vartheta_{x2}.\vartheta_{x_3},\ldots,\vartheta_{x_n}\}$ in Ω such that $F \leq I_\tau(\lambda,r)$, where $\lambda = \vee\{C_\tau(\vartheta_{x_i}) : i = 1,2,\ldots,n\}$ Since f is r-fuzzy weakly β-closed by Theorem 2.5 there exists a r-fuzzy β-open μ in Y with $f^{-1}(y) \leq f^{-1}(\mu) \leq C_\tau(X-\lambda)X - I_\tau(\lambda,r) \leq X - F.$. Therefore $y \in \mu$ and $\mu \wedge f(F) = \phi$. Thus $y \in Y - \beta C_\tau(f(F,r))$. This shows that $f(F)$ is r-fuzzy β-closed. Two non empty subsets λ and μ in X are strongly separated [16], if there exist open sets ϑ and η in X with $\lambda \leq \vartheta$ and $\mu \leq \eta$ and $C_\tau(\vartheta,r) \leq C_\tau(\eta,r) = \phi$. If λ and μ are singleton sets we may speak of points being strongly separated. We will use the fact that in a r-fuzzy normal space, disjoint closed sets are strongly separated. Recall that a space X is said to be r-fuzzy β-Hausdorff or in short $\beta - T_2$ [10], if for every pair of distinct points x and y, there exist two r-fuzzy β-open sets ϑ and η such that $x \in \vartheta$ and $y \in \eta$ and $\vartheta \wedge \eta = \phi$.

Theorem 3.12.

If $f:(X,\tau) \to (Y,\sigma)$ is a r-fuzzy weakly β-closed surjection and all pairs of disjoint r-fuzzy fibers are strongly separated, then Y is $\beta - T_2$.

Proof.

Let y and z be two points in Y. Let ϑ and η be open sets in X such that $f^{-1}(y) \in \vartheta$ and $f^{-1}(z) \in \eta$ respectively with $C_\tau(\vartheta) \wedge C_\tau(\eta) = \phi$. By r-fuzzy β-closedness (Theorem 3.4(v)) there are r-fuzzy β-open sets F and μ in Y such that $y \in F$ and $z \in \mu$, $f^{-1}(F) \leq C_\tau(\vartheta)$ and $f^{-1}(\mu) \leq C_\tau(\eta)$. Therefore $F \wedge \mu = \phi$, because $C_\tau(\vartheta) \wedge C_\tau(\eta) = \phi$ and f surjective. Then Y is $\beta - T_2$.

Corollary 3.13.

If $f:(X,\tau) \to (Y,\sigma)$ is r-fuzzy weakly β-closed surjection with all fibers closed and X is r-fuzzy normal, then Y is $\beta - T_2$.

Corollary 3.14.
If $f:(X,\tau)\to(Y,\sigma)$ is continuous r-fuzzy weakly β-closed surjection with X is a r-fuzzy compact T_2 space and Y is a r-fuzzy T_1 space, then Y is r-fuzzy compact $\beta-T_2$ space.

Proof.
Since f is a r-fuzzy continuous surjection and Y is a r-fuzzy T_1 space, Y is r-fuzzy compact and all fibers are r-fuzzy closed. Since X is r-fuzzy normal Y is also $\beta-T_2$.

Definition 3.15.
A topological space X is said to be r-fuzzy quasi H-closed (resp. r-fuzzy β-space), if every r-fuzzy open (resp. r-fuzzy β-closed) cover of X has a finite subfamily whose closures cover X. A subset λ of a r-fuzzy topological space X is r-fuzzy quasi H-closed relative to X (resp. r-fuzzy β-space relative to X) if every cover of λ by r-fuzzy open (resp. r-fuzzy β-closed) sets of X has a finite subfamily whose r-fuzzy closures cover λ.

Lemma 3.16.
A function $f:(X,\tau)\to(Y,\sigma)$ is open if and only if for each $\mu\le Y$, $f^{-1}(C_\tau(\mu,r))\le C_\tau(f^{-1}(\mu,r))$ [9].

Theorem 3.17.
Let X be an extremally disconnected space and $\beta O(X,\tau)$ closed under finite intersections. Let $f:(X,\tau)\to(Y,\sigma)$ be an open weakly β-closed function which is one-one and such that $f^{-1}(y)$ is quasi H-closed relative to X for each y in Y. If G is β-space relative to Y then $f^{-1}(G)$ is quasi H-closed.

Proof.
Let $\{\eta_\alpha:\alpha\in I\}$, (I being the index set) be an open cover of $f^{-1}(G)$. Then for each $y\in G\wedge f(X)$, $f^{-1}(y)\le\vee\{C_\tau(\eta_\alpha,r):\alpha\in I(y)\}=H_y$ for some finite subfamily $I(y)$ of I. Since X is extremally disconnected each $C_\tau(\eta_\alpha,r)$ is open, hence H_y is open in X. So by Corollary 3.10, there exists a β-closed set ϑ_y containing y such that $f^{-1}(\vartheta_y)\le C_\tau(H_y,r)$. Then, $\{\eta_y:y\in G\wedge f(X)\}\vee\{Y-f(X)\}$ is a β-closed cover of G, $G\le\vee\{C_\tau(\vartheta_y,r):y\in K\}\vee\{C_\tau(Y-f(X)\}$ for some finite subset K of $G\wedge f(X)$. Hence and by Lemma 3.16,

$$f^{-1}(G)\le\vee\{f^{-1}(C_\tau(\vartheta_y,r):y\in K\}\vee\{f^{-1}(C_\tau(Y-f(X)\}$$
$$\le\vee\{C_\tau(f^{-1}(\vartheta_y,r):y\in K\}\vee\{C_\tau(f^{-1}(Y-f(X)))\}\le\{C_\tau(f^{-1}(\vartheta_y,r)):y\in K\},$$

so $f^{-1}(G)\le\vee\{C_\tau(\eta_\alpha,r):\alpha\in I(y),y\in K\}$. Therefore $f^{-1}(G)$ is quasi H-closed.

Corollary 3.18.
Let $f:(X,\tau)\to(Y,\sigma)$ be as in Theorem 3.17. If Y is β-space, then X is quasi-H-closed.

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